A new boundary integral formulation to describe
three-dimensional motions of interfaces
between magnetic fluids

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Abstract

A new general three-dimensional hydrodynamic–magnetic boundary integral formulation for a magnetic free surfaces
in viscous flows at low Reynolds numbers is developed. The formulation is based on an extension of the Lorentz reciprocal
theorem for the incompressible flow of a magnetic fluid. Combining the reciprocal theorem and the fundamental solution
of a creeping flow we obtain the integral representation of the flow in terms of hydrodynamic and magnetic potentials.
According to this formulation, the magnetic and hydrodynamic quantities which are necessary for determination of the
dynamics of a magnetic liquid are established by means of appropriate integral equations at the boundary of the region
occupied by the magnetic liquid. The motion of a free surface with arbitrary magnetic properties and with the viscosity
of the magnetic liquid and the surrounding fluid not identical may be explored with the present formulation. Two relevant
physical parameters are revealed in the present hydrodynamic–magnetic boundary integral formulation: the ratio of the
magnetic permeability and the magnetic capillary number. The proposed boundary integral equations has been developed
in order to simulate the full time-dependent low Reynolds number distortion and orientation of a three-dimensional fer-
rofluid droplet under the action of shearing motions and magnetic fields.

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1. Introduction

It has become increasingly apparent in recent years that deformation of fluid interfaces under an applied
field has been the subject of numerous investigations. Applications include the breakup of rain drops in thunderstorms,
electrohydrodynamic atomization, the behavior of jets and drops in ink-jets plotters. A thorough review on drop deformation in arbitrary shear is given by Rallison [28] and Stone [31]. Magnetic drop deformation has been first studied by Arkhipenko et al. [2] and by Drozdova et al. [13]. Experiments carried out by

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Bacri et al. and Bacri and Salin [4,3,7] have shown that, when the magnetic field is increased and subsequently reduced, hysteresis in the drop deformation is observed. A related problem of a collapsing bubble in a magnetic fluid was studied by Cunha et al. [10].

The present work is concerned with a general three-dimensional boundary integral formulation to be used in numerical simulations for describing the deformation of a three-dimensional ferrofluid drop undergoing a magnetic field and a shear flow at low Reynolds numbers. The boundary integral formulation for a Stokes flow regime was first described, in a theoretical way, by Ladyzhenskaya [21] within the framework of hydrodynamic potentials. The boundary integral method was developed and implemented numerically by Youngreen and Acrivos [32] in a hydrodynamic problem of flow around an arbitrarily-shaped rigid particle. Rallison [26,27] applied the technique to study non-magnetic drop deformation and breakup in extensional and general linear flows. Many aspects related to integral equation methods for free-boundary problems are described in a book by Pozrikidis [25]. Boundary integral methods have been successfully used for simulations of potential flows around three-dimensional bodies [1], non-magnetic drop deformation and breakup [8,9], drop-to-drop interaction [22,15,12], characterization of non-magnetic emulsion rheology [23,33,24] and emulsion expansion and foam-drop dynamics [20,11,12].

Although most works have been concerned with axisymmetric or two-dimensional boundary integral formulation for magnetic drops in electric or magnetic field, which only require numerical treatment of line integrals, e.g. [30,5,6], the present work will consider the more difficult case of a three-dimensional integral formulation for a hydrodynamic–magnetic surface distortion. The problem falls naturally into two parts: that of finding the magnetic potential, and that of determining the fluid motion. We have combined the hydrodynamic and the magnetic problem by means of a boundary integral technique. The sections of this article are devoted to the mathematical development of a general three-dimensional boundary integral formulation for a deformable interface for a ferrofluid undergoing a shear flow in the presence of a magnetic field.

2. Governing equations

The theoretical formulation developed in this article may be applied for modeling the motion of deformable magnetic drops in an emulsion composed of drops of viscosity \( \eta' \), magnetic permeability \( \mu' \) and undisturbed radius \( a \) immersed in a second immiscible fluid of viscosity \( \eta \), magnetic permeability \( \mu \) with externally imposed velocity \( \mathbf{u}^\infty \) and magnetic \( \mathbf{H}^\infty \) fields, as described in Fig. 1. Such emulsions are called magneto-rheological emulsion. Hereupon, \( \lambda = \eta'/\eta \) and \( \pi = \mu'/\mu \) denote, respectively, the viscosity ratio and the magnetic permeability ratio between internal and external fluids. This problem is non-linear and gives rise to non-Newtonian effects assigned to magnetic stresses and the coupling between magnetism and hydrodynamics.

2.1. Magnetostatics

In the absence of an electrical field and if magnetic field does not vary with time, Maxwell’s equations [14,29] reduce to the magnetostatic limit with magnetic induction and magnetic field described by the following equations:

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} = 0,
\]

\[\text{(1)}\]

Fig. 1. Sketch of a magnetic fluid drop in an emulsion under an externally imposed velocity and magnetic fields.
where $\nabla$ denotes the partial differential operator, $B$ is the magnetic induction and $H$ the magnetic intensity vector. In addition, the magnetic relation,

$$B = \mu_0(M + H)$$

(2)

is valid at every point of the material. Here, $M$ is the local magnetization, that reveals the intrinsic polarization state of the continuum material generated by the magnetic field and $\mu_0 = 4\pi \times 10^{-7} \, \text{H} \cdot \text{m}^{-1}$ is the permeability of free space. The magnetizable liquid is assumed to obey a linear relation $M = \chi H$, with $\chi$ being the magnetic susceptibility. We focus therefore on dilute soft magnetic materials, i.e., superparamagnetic fluids, that have a very short memory, resulting in an instantaneous alignment of the particles with $H$. Under this condition, Eq. (2) reduces to $B = (K - 1)H = \mu H$, where $\mu = \mu_0(1 + \chi)$ denotes the permeability of the magnetic liquid and $K = \mu/\mu_0 = 1 + \chi$ is the relative permeability of the magnetic liquid. It is important to note that, in our formulation, we consider the magnetic permeability as being a magnetic material constant. Therefore, $\nabla \cdot H = 0$, i.e. $H$ is a solenoidal field. In addition, $H$ is an irrotational field, then $H = \nabla \phi$, where $\phi$ is the magnetic potential field, and the problem is governed by the Laplace equation $\nabla^2 \phi = 0$.

2.2. Hydrodynamics

As mentioned in Section 2, a hydrodynamic–magnetic coupled problem is studied here. Therefore, besides the magnetic equations, we must describe the hydrodynamic balance equations. In this sense, neglecting fluid inertia and compressibility, the hydrodynamic balance equations reduces to the Stokes flow regime described by Happel and Brenner [16] and Kim and Karrila [19],

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{\sigma} = 0,$$

(3)

where $\mathbf{u}$ and $\mathbf{\sigma}$ represents the Eulerian velocity field and the stress tensor of the fluid. In a ferrohydrodynamic context, as the one explored in this article, the coupling between magnetism and hydrodynamics is given by the stress tensor $\mathbf{\sigma}$ that considers magnetic effects on the flow, namely,

$$\mathbf{\sigma} = -P \mathbf{I} + 2\eta \mathbf{D} + \mathbf{B} \mathbf{H},$$

(4)

where the notation $\mathbf{B} \mathbf{H}$ corresponds to the dyadic or tensorial product between the $\mathbf{B}$ and $\mathbf{H}$ usually written as $\mathbf{B} \otimes \mathbf{H}$, $\eta$ denotes the fluid shear viscosity, $\mathbf{I}$ is the identity tensor, $P$ is the total pressure, namely

$$P = p_h + p_m,$$

(5)

where $p_h$ is the hydrodynamic pressure and $p_m = \mu_0 (\mathbf{H} \cdot \mathbf{H})/2$ is the magnetic pressure. Furthermore, $\mathbf{D} = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ denotes the rate of strain tensor and $(\nabla \mathbf{u})^T$ denotes the transpose tensor of $\nabla \mathbf{u}$.

3. Boundary conditions

On a drop interface $S$ with surface tension $\Gamma$, the boundary conditions require a continuous velocity across the interface and a balance between the net surface traction and the surface forces, that express the discontinuity at the interface. Moreover, the magnetostatic regime states that the normal components of $\mathbf{B}$ and the tangential components of $\mathbf{H}$ are continuous on the interface between different media (at all points across the interface). Mathematically, these conditions are expressed as

$$\mathbf{u} \to \mathbf{u}^\infty \text{ as } |x| \to \infty; \quad \mathbf{u}(x) = \mathbf{u}'(x) \quad \forall x \in S$$

(6)

and

$$n \cdot \mathbf{B}'(x) = n \cdot \mathbf{B}(x) \quad \text{and} \quad n \times \mathbf{H}'(x) = n \times \mathbf{H}(x) \quad \forall x \in S,$$

(7)

where $\mathbf{u}'$, $\mathbf{H}'$ and $\mathbf{B}'$ denote, respectively, the flow, the magnetic field and the magnetic induction inside the drop and $n$ is the outwards unit vector normal to $S$. Considering an interface free of surface viscosity, surface elasticity and surface module of bending and dilatation, the constitutive equation for the traction jump $\Delta t = [n \cdot \mathbf{\sigma}]$ is written as [25]

$$\Delta t = [n \cdot \mathbf{\sigma}] = \Gamma (\nabla^5 \cdot n)n - (I - nn) \cdot \nabla \Gamma;$$

(8)
The notation $[[\ ]]$ denotes a jump in flow quantities, $t = n \cdot \sigma$ is the surface traction, $\nabla^S = (I - nn)$ denotes the gradient operation tangent to the interface, consequently $\nabla^S \cdot n$ denotes twice the mean curvature $\kappa$ of the interface. The traction jump given by Eq. (8) could be seen as the summation of a normal traction jump $nn \cdot \Delta t$ and a tangential stress jump $(I - nn) \cdot \Delta t$. The normal component includes the effect promoted by the surface tension $C$ while the tangential component is that due to interfacial tension gradients, associated with the presence of surfactants in the fluid, named Marangoni effect. For clean interfaces the Marangoni stresses are ignored.

Furthermore, using a Lagrangian representation for the interface evolution of a drop, one gets a kinematic constraint relating changes in the interface position to the local velocity,

$$\frac{Dx}{Dt} = u(x) \quad \forall x \in S,$$

where $D/Dt$ denotes the well-known material derivative.

4. Magnetic boundary integral formulation

In this section we present a three-dimensional boundary integral representation for Laplace equation, resulting from the magnetostatic conditions given in Eq. (1), in terms of singularities at the interface between two magnetic fluids.

4.1. Reciprocal theorem for a magnetic potential

Consider a closed region of fluid $V$ bounded by a surface $S$. Following this assumption, consider two distinct magnetic potential fields $\phi$ and $\phi_i$ acting, respectively, over two different magnetic fluids. According to the Green’s second identity [17,18], we have

$$\int_S (\phi \nabla \phi_i - \phi_i \nabla \phi) \cdot n \, dS = \int_V (\phi \nabla^2 \phi_i - \phi_i \nabla^2 \phi) \, dv,$$

(10)

where $\phi$ and $\phi_i$ are two scalar functions of position. Being $\phi$ and $\phi_i$ harmonic functions, such as the magnetic potential, the RHS (right-hand side) of Eq. (10) vanishes, leading to the reciprocal theorem for harmonic functions,

$$\int_S \phi \nabla \phi_i \cdot n \, dS = \int_S \phi_i \nabla \phi \cdot n \, dS.$$

(11)

Eq. (11) expresses that, if the fundamental solution for the magnetic potential field $\phi_i$ is known, any field of interest $\phi$ may be determined in terms of singularity distributions on the interface $S$.

4.2. Integral representation for a magnetic potential

Now, let us consider the integral representation solution for a magnetic potential field $\phi$. Here, the known magnetic potential field $\phi_i$ corresponds to the fundamental solution of $\nabla^2 \phi_i = h \delta(r)$ that is given by

$$\phi_i(r) = \frac{h}{4\pi \mu_0 r} = \frac{h}{4\pi \mu_0} \mathcal{C}(r),$$

(12)

and, consequently,

$$\nabla \phi_i(r) = - \frac{hr}{4\pi \mu_0 r^3} = \frac{h}{4\pi \mu_0} \nabla \mathcal{C}(r),$$

(13)

where $\mathcal{C}(r) = 1/r$ is the free space Green’s function corresponding to a source point and $\nabla \mathcal{C}(r) = -r/r^3$ denotes a potential dipole. The solution in Eq. (12) corresponds to the magnetic potential field due to a source point with strength $h$. Here, $r = x - x_0$, with $x$ being an arbitrary point of the domain $V$, and $x_0$ the location of the magnetic dipole (singularity) and $r = |r|$.
Now, we apply the fundamental solution given in Eq. (12) to the reciprocal theorem given by Eq. (11). Here, $\phi$ is the unknown potential in the domain $V$ and $\phi_i$ is the potential of a point source, that is singular as $r \to 0$. Two possibilities relative to the singularity location in the domain of the flow are considered below.

4.2.1. Singularity outside $V$

In this case $\delta(r) = 0$ inside $V$. After discarding the arbitrary constant $h \neq 0$, the reciprocal theorem in Eq. (11) reduces to

$$
\int_S [\phi(x) \nabla \phi_i(r) - \phi_i(r) \nabla \phi(x)] \cdot \mathbf{n} \, dS = 0.
$$

(14)

Here, the potential $\phi$ is not singular inside $V$, because $x_0$ is outside $V$.

4.2.2. Singularity inside $V$

Consider the case in which a singularity is placed at $x_0$ inside $V$. The singularity needs to be excluded from the region of integration. To overcome this problem, a small spherical boundary of radius $\varepsilon$ and volume $V_\varepsilon$ centered at $x_0$ involving this singularity is considered as shown in Fig. 2. Here, $\mathbf{n}_i$ and $\mathbf{n}$ mean, respectively, the inwards and outwards unit vector normal to the surface $S$, then $\mathbf{n} = -\mathbf{n}_i$. Then, outside the small sphere, throughout the remaining volume $V - V_\varepsilon$ the functions within the square brackets in Eq. (11) are continuous.

In this way, again discarding the arbitrary constant $h$, the reciprocal theorem Eq. (11), applied to the surface $S - S_\varepsilon$ that bounds the volume $V - V_\varepsilon$, becomes

$$
\int_S [\phi_i(r) \nabla \phi(x) - \phi(x) \nabla \phi_i(r)] \cdot \mathbf{n}_i \, dS + \int_{S_\varepsilon} [\phi_i(r) \nabla \phi(x) - \phi(x) \nabla \phi_i(r)] \cdot \mathbf{n} \, dS = 0.
$$

(15)

Now, consider the integral over $S_\varepsilon$ containing the singularity $x_0$, with $dS_\varepsilon = \varepsilon^2 \, d\Omega$ and $d\Omega$ being the infinitesimal solid angle. Based on the fundamental solution given in (12) and (13), the expressions for the potential monopole $\phi_i(r)$ and the potential dipole $\nabla \phi_i(r)$ inside $S_\varepsilon$, with the inwards unit normal vector being $\mathbf{n}_i = r/\varepsilon$, are given by

$$
\phi_i(r) \approx \frac{1}{\varepsilon} \quad \text{and} \quad \nabla \phi_i(r) \approx -\frac{r}{\varepsilon^3} = -\mathbf{n}_i/\varepsilon^2.
$$

(16)

Therefore, for the limit $\varepsilon \to 0$, one obtains

$$
\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \phi_i(r) \nabla \phi(x) \cdot \mathbf{n}_i \, dS = \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{1}{\varepsilon} \nabla \phi(x) \cdot \mathbf{n}_i \varepsilon^2 \, d\Omega = \phi_i(\mathbf{r}) \to 0
$$

(17)

and

$$
\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \phi(x) \nabla \phi_i(r) \cdot \mathbf{n}_i \, dS = -\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \phi(x) \frac{1}{\varepsilon^2} \varepsilon^2 \, d\Omega = -\phi(x_0).
$$

(18)

![Fig. 2. Fluid domain $V$ bounded by a surface $S$ broken down into $V_\varepsilon$ and $V - V_\varepsilon$.](image-url)
With the results given by Eqs. (17) and (18), Eq. (15) reduces to
\[ \phi(x_0) = -\int_S [\phi(x)\nabla\mathcal{G}(r) - \mathcal{G}(r)\nabla\phi(x)] \cdot n\,dS. \tag{19} \]

By analogy with corresponding results in the theory of electrostatics [25] and elastostatics [17], the two integrals on the RHS of Eq. (19) are termed single-layer and double-layer potentials. They represent, respectively, a boundary distribution of the Green’s functions \( \mathcal{G}(r) \) and \( \nabla\mathcal{G}(r) \), amounting to boundary distributions of magnetic source points and magnetic point dipoles.

4.3. Integral representation in terms of jump conditions

Now we are interested in the solution of the magnetic potential at the interface \( S \), which may be found by the application of the jump condition \( [\phi(x_0) + \phi'(x_0)]/2 \) using Eqs. (22) and (25). Limiting \( x_0 \) to the interface, and

4.3.1. Singularity inside the external fluid domain \( V \)

According to the reciprocal theorem in (14), for the external fluid \( \phi' \) (inside the particle) with the point \( x_0 \) exterior to the particle, we obtain
\[ \int_S [\phi'(x)\nabla\mathcal{G}(r) - \mathcal{G}(r)\nabla\phi'(x)] \cdot n\,dS = 0. \tag{20} \]

Now, applying Eq. (19) for the external fluid under an imposed \( \phi^\infty(x_0) \) one obtains
\[ \phi(x_0) = -\int_S [\phi(x)\nabla\mathcal{G}(r) - \mathcal{G}(r)\nabla\phi(x)] \cdot n'\,dS + \phi^\infty(x_0) \)
and, subtracting Eq. (20) of (21), one obtains in terms of the jump condition \( \phi(x) - \phi'(x) \) and \( \nabla[\phi(x) - \phi'(x)] \) the following integral representation,
\[ \phi(x_0) = \phi^\infty(x_0) + \int_S [\phi(x) - \phi'(x)]\nabla\mathcal{G}(r) \cdot n\,dS - \int_S \mathcal{G}(r)\nabla[\phi(x) - \phi'(x)] \cdot n\,dS. \tag{22} \]

4.3.2. Singularity inside the internal fluid domain \( V' \)

By analogy with the procedure of deriving the integral representation for the external fluid, we determine the integral representation for the internal fluid applying Eq. (19) as being,
\[ \phi'(x_0) = -\int_S [\phi'(x)\nabla\mathcal{G}(r) - \mathcal{G}(r)\nabla\phi'(x)] \cdot n\,dS. \tag{23} \]

Moreover, using the reciprocal identity (14) for the external fluid \( \phi \) (outside the particle) with a point \( x_0 \) that is located in the interior of the particle, can be obtained
\[ \int_S [\phi(x)\nabla\mathcal{G}(r) - \mathcal{G}(r)\nabla\phi(x)] \cdot n'\,dS - \phi^\infty(x_0) = 0 \]
and, subtracting this result of Eq. (23), one obtains
\[ \phi'(x_0) = \phi^\infty(x_0) + \int_S [\phi(x) - \phi'(x)]\nabla\mathcal{G}(r) \cdot n\,dS - \int_S \mathcal{G}(r)\nabla[\phi(x) - \phi'(x)] \cdot n\,dS. \tag{25} \]

4.4. Integral representation at the interface for a magnetic potential

Now, we are interested in the solution of the magnetic potential at the interface \( S \) that may be found by the application of the jump condition \( [\phi(x_0) + \phi'(x_0)]/2 \) using Eqs. (22) and (25). Limiting \( x_0 \) to the interface, and
taking into account the boundary conditions \( \phi(x_0) = \phi'(x_0) \), \( \phi(x) = \phi'(x) \) and \( \mu \nabla \phi(x) \cdot n = \mu' \nabla \phi'(x) \cdot n \), the magnetic potential integral representation at the interface \( S \) of two magnetic fluids is given by

\[
\phi(x_0) = \phi^∞(x_0) + \left( \frac{1 - x}{\alpha} \right) \int_S \mathcal{G}(r) \nabla \phi(x) \cdot n \, dS. \tag{26}
\]

Eq. (26) gives the magnetic potential for each point \( x_0 \) at the interface in terms of a distribution of singularities (i.e. source points and dipoles), representing the magnetic potential disturbances induced by the neighbouring points over the surface, located at points \( x \).

5. Hydrodynamic boundary integral formulation

In this section, a boundary integral formulation to compute the Stokes flow of a magnetic fluid is derived by solving integral equations for functions that are evaluated over the boundaries. This formulation couples the integral equations for the velocity and the magnetic potential fields.

5.1. Reciprocal theorem for the flow of a magnetic fluid

Consider a closed region of fluid \( V \) bounded by a surface \( S \). Then consider two unrelated incompressible flows of two different magnetic fluids with densities \( \rho \) and \( \rho_i \), viscosities \( \eta \) and \( \eta_i \), magnetic permeabilities \( \mu \) and \( \mu_i \) and stress fields \( \sigma \) and \( \sigma_i \), respectively.

Flow 1: \( \mathbf{u}, \mathbf{H}, \sigma (\rho, \eta, \mu) \). The balance equations for mass and momentum are, respectively,

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \sigma = 0. \tag{27}
\]

Here, locally, \( \mathbf{u} \) is the Eulerian velocity and, \( \sigma \) is the stress field. As mentioned before the constitutive equation for a magnetic fluid is given by

\[
\sigma = -P I + 2\eta \mathbf{D} + \mu \mathbf{HH}, \tag{28}
\]

where \( I \) is the identity tensor and \( \mathbf{D} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2 \) is the rate of strain tensor. Again, \( P = p_h + p_m \), where \( p_h \) is the hydrodynamic pressure and \( p_m = \mu_0 H^2/2 \) is the magnetic pressure.

Flow 2: \( \mathbf{u}_i, \mathbf{H}_i, \sigma_i (\rho_i, \eta_i, \mu_i) \). Similarly to Eq. (27), the balance equations for mass and momentum and the constitutive equation for this flow are, respectively,

\[
\nabla \cdot \mathbf{u}_i = 0, \quad \nabla \cdot \sigma_i = 0 \tag{29}
\]

and

\[
\sigma_i = -P_i I + 2\eta_i \mathbf{D}_i + \mu_i \mathbf{H}_i \mathbf{H}_i, \tag{30}
\]

where \( \mathbf{D}_i = [\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T]/2 \) and \( P_i \) are the rate of strain tensor and the total pressure field, respectively. Furthermore, remind that the following tensorial operation for an incompressible fluid is valid \( I : \mathbf{D} = \nabla \cdot \mathbf{u} = 0 \) and \( I : \mathbf{D}_i = \nabla \cdot \mathbf{u}_i = 0 \), one may obtain

\[
\sigma : \mathbf{D}_i = 2\eta D : \mathbf{D}_i + \mu \mathbf{HH} : \mathbf{D}_i \tag{31}
\]

and, similarly,

\[
\sigma_i : \mathbf{D} = 2\eta_i D : \mathbf{D} + \mu_i \mathbf{H} \mathbf{H}_i : \mathbf{D}. \tag{32}
\]

The symmetry of both \( \mathbf{D} \) and \( \mathbf{D}_i \) requires that \( \mathbf{D} : \mathbf{D}_i = \mathbf{D}_i : \mathbf{D} \). Using this argument, Eq. (31) becomes

\[
\mathbf{D} : \mathbf{D}_i = \mathbf{D}_i : \mathbf{D} = \frac{1}{2\eta} (\sigma : \mathbf{D}_i - \mu \mathbf{HH} : \mathbf{D}_i), \tag{33}
\]

and, substituting the result (33) into Eq. (32), we obtain

\[
\sigma_i : \mathbf{D} = \frac{\eta_i}{\eta} \sigma : \mathbf{D}_i - \mu \left( \frac{\eta_i}{\eta} \mathbf{HH} : \mathbf{D}_i - \frac{\mu_i}{\mu} \mathbf{H} \mathbf{H}_i : \mathbf{D} \right). \tag{34}
\]
It should be important to note that, for superparamagnetic fluids is valid to suppose that \( \mu_0 \mathbf{M} \times \mathbf{H} = 0 \), no internal torques take place in the fluid and the stress tensor is symmetric. Consequently,
\[
\sigma : \mathbf{D}_i = \sigma : \nabla \mathbf{u}_i = \nabla \cdot (\mathbf{u}_i \cdot \sigma) - \mathbf{u}_i \cdot (\nabla \cdot \sigma). \tag{35}
\]

Now, applying Cauchy’s Eq. (27) to the last term in the RHS of Eq. (35). Then, Eq. (35) reduces to
\[
\sigma : \mathbf{D}_i = \nabla \cdot (\mathbf{u}_i \cdot \sigma). \tag{36}
\]
Similarly, one may obtain that
\[
\sigma_i : \mathbf{D} = \nabla \cdot (\mathbf{u} \cdot \sigma_i). \tag{37}
\]

Thereafter, we can evaluate the term \( \mathbf{H} \cdot \mathbf{D}_i \). Note that \( \mathbf{H} \) is a second rank symmetric tensor. Therefore
\[
\mathbf{H} : \mathbf{D}_i = \mathbf{H} \cdot \nabla \mathbf{u}_i = \nabla \cdot (\mathbf{u}_i \cdot \mathbf{H}) - \mathbf{u}_i \cdot \nabla \cdot (\mathbf{HH}). \tag{38}
\]

Using a vectorial identity, the magnetostatic regime balance equations \( \nabla \times \mathbf{H} = 0 \) and \( \nabla \cdot \mathbf{B} = 0 \) and the assumption of a constant magnetic susceptibility, that results in \( \nabla \cdot \mathbf{H} \). In this way, one obtains that
\[
\nabla \cdot (\mathbf{HH}) = \mathbf{H} \cdot \nabla \mathbf{H} + \mathbf{H} (\nabla \cdot \mathbf{H}) = \nabla \left( \frac{H^2}{2} \right) - \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{H} (\nabla \cdot \mathbf{H}) = \nabla \left( \frac{H^2}{2} \right) \tag{39}\]

Then, substituting (39) into (38), results in:
\[
\mathbf{H} : \mathbf{D}_i = \nabla \cdot (\mathbf{u}_i \cdot \mathbf{HH}) - \mathbf{u}_i \cdot \nabla \left( \frac{H^2}{2} \right). \tag{40}\]

If the same steps are applied to the term \( \mathbf{H}_i : \mathbf{D} \), it must reduce in an analogous fashion to
\[
\mathbf{H}_i \cdot \mathbf{H}_i : \mathbf{D} = \nabla \cdot (\mathbf{u} \cdot \mathbf{H}_i \mathbf{H}_i) - \mathbf{u} \cdot \nabla \left( \frac{H^2}{2} \right). \tag{41}\]

Now, substituting Eqs. (36), (37), (40) and (41) into Eq. (34), we found that
\[
\nabla \cdot (\mathbf{u} \cdot \sigma_i) = \frac{\eta}{\eta} \nabla \cdot (\mathbf{u}_i \cdot \sigma) - \mu \left\{ \frac{\eta}{\eta} \left[ \nabla \cdot (\mathbf{u}_i \cdot \mathbf{HH}) - \mathbf{u}_i \cdot \nabla \left( \frac{H^2}{2} \right) \right] - \frac{\mu_i}{\mu} \left[ \nabla \cdot (\mathbf{u} \cdot \mathbf{H}_i \mathbf{H}_i) - \mathbf{u} \cdot \nabla \left( \frac{H^2}{2} \right) \right] \right\}. \tag{42}\]

Finally, after making a few algebraic manipulations, we obtain the expression for the generalized Lorentz reciprocal theorem for a Stokes flow of a magnetic fluid,
\[
\eta \nabla \cdot (\mathbf{u} \cdot \sigma_i) - \eta \nabla \cdot (\mathbf{u}_i \cdot \sigma) = \mu \left[ \nabla \cdot (\mathbf{u} \cdot \mathbf{H}_i \mathbf{H}_i) - \mathbf{u} \cdot \nabla \left( \frac{H^2}{2} \right) \right] - \mu \left[ \nabla \cdot (\mathbf{u}_i \cdot \mathbf{HH}) - \mathbf{u}_i \cdot \nabla \left( \frac{H^2}{2} \right) \right]. \tag{43}\]

5.2. Integral representation for a creeping flow of a magnetic fluid

Consider a particular flow of interest with velocity \( \mathbf{u} \), magnetic field \( \mathbf{H} \) and stress tensor \( \sigma \). The known flow is the one due to a point force of strength \( f \) and located at a point \( x_0 \). Suppose that the inertia of both fluids has a negligible influence on the motion of the fluid elements, and by convenience take \( \eta = \eta_i, \rho = \rho_i, \mu = \mu_i \) and \( \mathbf{H}_i = \mathbf{0} \). Flow 1 and flow 2 for this particular situation are described as follows:

**Flow 1: \( \mathbf{u}, \mathbf{H}, \sigma \).** The equations for conservation of mass and momentum for the flow 1 are, respectively,
\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \sigma = 0. \tag{44}\]

As stated before, the constitutive equation for a magnetic fluid is given by
\[
\sigma = -p \mathbf{I} + 2\eta \mathbf{D} + \mu \mathbf{HH}. \tag{45}\]
Flow 2: \( \mathbf{u}_i, \sigma_i \). The fundamental solution for Stokes equations corresponds to the velocity and stress fields at a point \( \mathbf{x} \) produced by a point force \( \mathbf{h} \) located at \( \mathbf{x}_0 \),

\[
\nabla \cdot \mathbf{u}_i = 0, \quad \nabla \cdot \sigma_i = -h \delta(\mathbf{x} - \mathbf{x}_0),
\]

with \( |\mathbf{u}_i| \to 0 \) and \( |\sigma_i| \to 0 \) as \( |\mathbf{x}| \to \infty \). The fundamental solution is derived in a straightforward way by using Fourier transforms,

\[
\mathbf{u}_i(x) = \frac{f}{8\pi \eta} \cdot \mathbf{G}(r); \quad \sigma_i(x) = -\frac{3f}{4\pi} \cdot \mathbf{T}(r),
\]

where the stokeslet, \( \mathbf{G} \) and the stresslet \( \mathbf{T} \) are defined, respectively, by the following expressions:

\[
\mathbf{G}(r) = \frac{I}{r} + \frac{rr}{r^3}; \quad \mathbf{T}(r) = \frac{rr}{r^3}.
\]

The above functions are the kernels of the free-space Green’s functions that map the force \( \mathbf{f} \) at \( \mathbf{x}_0 \) to the fields at \( \mathbf{x} \) in an unbounded three-dimensional domain. Here, \( r = \mathbf{x} - \mathbf{x}_0 \), and \( r = |\mathbf{r}| \). Physically, \( \mathbf{u} = \mathbf{G}(r) \cdot \mathbf{f} \) expresses the velocity field due to a concentrated point force \( \mathbf{f}(\mathbf{r}) \) placed at the point \( \mathbf{x}_0 \), and may be seen as the flow produced by the slow settling motion of a point particle. \( \mathbf{T} \) is the stress tensor associated with the Green’s function \( \mathbf{G}_{ij} \) and \( \sigma_{ik}(x) = \mathbf{T}_{ijk} f_j \) is a fundamental solution of the Stokes produced by the hydrodynamic dipole \( \mathbf{G} \cdot \nabla \delta(\mathbf{r}) \). Note that \( \mathbf{T}_{ijk} = \mathbf{T}_{jik} \) as required by symmetry of the stress tensor \( \sigma \). As before, note that the notation \( rr \) corresponds to the dyadic or tensorial product between the \( \mathbf{r} \) and \( \mathbf{r} \).

At this point, substituting the expressions of the point-force solution (47) into (43) and discarding the arbitrary constant \( \mathbf{f} \) one obtains

\[
-\frac{3}{4\pi} \nabla \cdot [\mathbf{u}(x) \cdot \mathbf{T}(r)] - \frac{1}{8\pi \eta} \nabla \cdot [\mathbf{G}(r) \cdot \sigma(x)] = -\frac{\mu}{8\pi \eta} \left\{ \nabla \cdot [\mathbf{G}(r) \cdot \mathbf{HH}(x)] - \mathbf{G}(r) \cdot \nabla \left( \frac{H^2(x)}{2} \right) \right\}.
\]

Now, Eq. (49) can still be simplified by the result of the incompressibility condition \( \nabla \cdot \mathbf{G} = 0 \) and the symmetry of \( \mathbf{G} \) tensor, so that \( \mathbf{G}(r) \cdot \nabla (H^2/2) = \nabla \cdot [\mathbf{G}(r)(H^2/2)] \). Under these conditions, Eq. (49) becomes

\[
-\frac{3}{4\pi} \nabla \cdot [\mathbf{u}(x) \cdot \mathbf{T}(r)] - \frac{1}{8\pi \eta} \nabla \cdot [\mathbf{G}(r) \cdot \sigma(x)] = -\frac{\mu}{8\pi \eta} \nabla \cdot \left\{ \mathbf{G}(r) \cdot \left[ \mathbf{HH}(x) - \left( \frac{H^2(x)}{2} \right) I \right] \right\}.
\]

It is important to note that the above equation is valid everywhere except at the singular point \( \mathbf{x}_0 \). Now, consider a material volume of fluid \( V \) bounded by a simply or multiply connected surface \( S \) in order to evaluate the integration of Eq. (50). The surface \( S \) may be composed of fluid surfaces, fluid interfaces or solid surfaces. As before, there are two situations to be considered next.

5.2.1. Singularity outside \( V \)

For this case, we select the singularity point \( \mathbf{x}_0 \) outside \( V \). Then, all terms of the reciprocal theorem are regular throughout \( V \), and thus after integrating Eq. (50) the integral representation of the reciprocal theorem takes the form

\[
-\frac{3}{4\pi} \int_V \nabla \cdot [\mathbf{u}(\mathbf{x}) \cdot \mathbf{T}(r)]dV - \frac{1}{8\pi \eta} \int_V \nabla \cdot [\mathbf{G}(r) \cdot \sigma(x)]dV
\]

\[
= -\frac{\mu}{8\pi \eta} \int_V \nabla \cdot \left\{ \mathbf{G}(r) \cdot \left[ \mathbf{HH}(x) - \left( \frac{H^2(x)}{2} \right) I \right] \right\}dV.
\]

Besides, the volume integrals in Eq. (51) are converted to the surface integrals over \( S \), by using the divergence theorem. So,

\[
-\frac{1}{8\pi \eta} \int_S \mathbf{G}(r) \cdot \sigma(x) \cdot \mathbf{n}(x)dS - \frac{3}{4\pi} \int_S \mathbf{u}(x) \cdot \mathbf{T}(r) \cdot \mathbf{n}(x)dS
\]

\[
+ \frac{\mu}{8\pi \eta} \int_S \mathbf{G}(r) \cdot \left[ \mathbf{HH}(x) - \left( \frac{H^2(x)}{2} \right) I \right] \cdot \mathbf{n}(x)dS = 0,
\]
where \( \mathbf{n} \) is the outwards unit vector normal to the surface \( S \). Eq. (52) is the integral representation of the flow if the singularity point is outside \( V \).

5.2.2. Singularity inside \( V \)

Similar to the analysis presented in Section §4.2, if a singularity located at \( x_0 \) into \( V \), it is necessary to exclude it of our integration step over \( V \). We repeat the same procedure before defining a small spherical region \( V_e \) (i.e. the singularity cut-off) of radius \( \varepsilon \) centered at \( x_0 \), as illustrated in Fig. 2. In addition, the functions into Eq. (50) are regular throughout the reduced volume \( V - V_e \). Then, integrating the Eq. (50) over \( V - V_e \) and converting the volume integral into a surface integral using the divergence theorem, we have

\[
- \frac{1}{8\pi\eta} \int_{S_e} \mathbf{G}(r) \cdot \mathbf{\sigma}(x) \cdot \mathbf{n}(x) dS - \frac{3}{4\pi} \int_{S_e} \mathbf{u}(x) \cdot \mathbf{T}(r) \cdot \mathbf{n}(x) dS \\
+ \frac{\mu}{8\pi\eta} \int_{S_e} \mathbf{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] \cdot \mathbf{n}(x) dS = 0,
\]

(53)

where \( S_e \) is the spherical surface enclosing \( V_e \), as indicated in Fig. 2. Making the radius \( \varepsilon \) tending to zero, we obtain the following expressions for the leading order terms in \( \varepsilon \) for the tensors \( \mathbf{G} \) and \( \mathbf{T} \), remembering that \( \mathbf{n}_i = \mathbf{r}/\varepsilon \),

\[
\mathbf{G}(r) \approx \frac{I}{\varepsilon} + \frac{rr}{\varepsilon^3} = \frac{I}{\varepsilon} + \frac{n_i n_i}{\varepsilon^3}; \quad \mathbf{T}(r) \approx \frac{rrr}{\varepsilon^5} = \frac{n_i n_i n_i}{\varepsilon^5}.
\]

(54)

Over \( S_e \), \( dS = d^2\Omega \), where, as before, \( \Omega \) is the differential solid angle. Substituting these expressions along with Eq. (53) and taking the limit \( \varepsilon \to 0 \) we obtain

\[
\lim_{\varepsilon \to 0} \int_{S_e} \mathbf{G}(r) \cdot \mathbf{\sigma}(x) \cdot \mathbf{n}(x) dS = \lim_{\varepsilon \to 0} \int_{S_e} \left( \frac{I}{\varepsilon} + \frac{rr}{\varepsilon^3} \right) \cdot \mathbf{\sigma}(x) \cdot \mathbf{n}(x) \varepsilon^2 d\Omega \\
= \lim_{\varepsilon \to 0} \int_{S_e} \left( \frac{I}{\varepsilon} + \frac{n_i n_i}{\varepsilon^3} \right) \cdot \mathbf{\sigma}(x) \cdot \mathbf{n}(x) \varepsilon^2 d\Omega = \mathcal{C}(\varepsilon) \to 0.
\]

(55)

As \( \varepsilon \to 0 \), the values of \( \mathbf{u}, \mathbf{H} \) and \( \mathbf{\sigma} \) tend to their corresponding values at the center of \( V_e \), i.e. to \( \mathbf{u}(x_0) \), \( \mathbf{H}(x_0) \) and \( \mathbf{\sigma}(x_0) \), respectively. Due to the same dependence on \( \varepsilon \) (decreasing linearly), the following term tends also to zero in the limit \( \varepsilon \to 0 \),

\[
\lim_{\varepsilon \to 0} \int_{S_e} \mathbf{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] \cdot \mathbf{n}(x) \varepsilon^2 d\Omega = \mathcal{C}(\varepsilon) \to 0.
\]

(56)

Also, the stresslet integral contribution to the flow is evaluated. At the limit \( \varepsilon \to 0 \),

\[
\lim_{\varepsilon \to 0} \int_{S_e} \mathbf{u}(x) \cdot \mathbf{T}(r) \cdot \mathbf{n}(x) dS = \lim_{\varepsilon \to 0} \int_{S_e} \mathbf{u}(x) \cdot \left( \frac{rrr}{\varepsilon^5} \right) \cdot \mathbf{n}(x) dS \\
= \lim_{\varepsilon \to 0} \int_{S_e} \mathbf{u}(x) \cdot \frac{rrr}{\varepsilon^5} \cdot \mathbf{n}(x) dS = \frac{\mathbf{u}(x_0)}{\varepsilon^4} \cdot \int_{S_e} r r dS.
\]

(57)

And, using the divergence theorem,

\[
\int_{S_e} r r dS = \varepsilon \int_{S_e} r n_i dS = \varepsilon \int_{V_e} \nabla r dV = \frac{4\pi}{3} \mathbf{I} \varepsilon^4.
\]

(58)

Therefore, substituting (55), (56) and (57) into Eq. (53), we have

\[
\mathbf{u}(x_0) = \frac{1}{8\pi\eta} \int_{S} \mathbf{G}(r) \cdot \mathbf{\sigma}(x) \cdot \mathbf{n}(x) dS + \frac{3}{4\pi} \int_{S} \mathbf{u}(x) \cdot \mathbf{T}(r) \cdot \mathbf{n}(x) dS \\
- \frac{\mu}{8\pi\eta} \int_{S} \mathbf{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] \cdot \mathbf{n}(x) dS,
\]

(59)

Eq. (59) is the integral representation for the Stokes flow of a magnetic fluid in terms of boundary distributions involving the Green’s functions \( \mathbf{G} \) and the stresslet \( \mathbf{T} \). The first distribution on the RHS of (59) is termed the
single-layer potential, and the second distribution is termed the double-layer potential. Both integrals have already appeared in three-dimensional boundary integral formulations of non-magnetic fluids. The last integral however represents an extra single-layer potential contribution by the fact that the fluid is polar.

5.3. Integral representation in terms of the traction jump

5.3.1. Singularity inside the external fluid domain \( V \)

Using the reciprocal identity (52) for the internal flow \( u' \) (inside the particle) with the point \( x_0 \) located exterior to the particle, results

\[
\begin{align*}
- \frac{1}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \sigma'(x) \cdot n(x) \, dS & - \frac{3\lambda}{4\pi} \int_S u'(x) \cdot \mathcal{T}(r) \cdot n(x) \, dS \\
+ \frac{\alpha \mu}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \left[ H' H'(x) - \left( \frac{H'^2(x)}{2} \right) I \right] \cdot n(x) \, dS &= 0,
\end{align*}
\]

where as defined before, \( \lambda = \eta' / \eta \) and \( \alpha = \mu' / \mu \). Now, applying Eq. (59) to the external flow undergoing an ambient flow \( u^\infty(x_0) \),

\[
\begin{align*}
u(x_0) &= u^\infty(x_0) + \frac{1}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \sigma'(x) \cdot n'(x) \, dS + \frac{3\lambda}{4\pi} \int_S u'(x) \cdot \mathcal{T}(r) \cdot n'(x) \, dS \\
- \frac{\alpha \mu}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \left[ H' H'(x) - \left( \frac{H'^2(x)}{2} \right) I \right] \cdot n'(x) \, dS,
\end{align*}
\]

and combining the result with Eq. (60), the integral representation is obtained in terms of the traction jump \( \Delta t(x) = [\sigma(x) - \sigma'(x)] \cdot n(x) \),

\[
\begin{align*}
u(x_0) &= u^\infty(x_0) - \frac{1}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \Delta t(x) \, dS - \frac{3\lambda}{4\pi} \int_S [u(x) - \lambda u'(x)] \cdot \mathcal{T}(r) \cdot n(x) \, dS \\
+ \frac{\mu}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] - \alpha \left[ HH'(x) - \left( \frac{H'^2(x)}{2} \right) I \right] \cdot n(x) \, dS.
\end{align*}
\]

5.3.2. Singularity inside the internal fluid domain \( V' \)

We repeat the above calculation for the internal flow. Hence, the integral representation of this flow is obtained when Eq. (59) is applied,

\[
\begin{align*}
u'(x_0) &= \frac{1}{8\pi \lambda \eta} \int_S \mathcal{G}(r) \cdot \sigma'(x) \cdot n(x) \, dS + \frac{3}{4\pi} \int_S u'(x) \cdot \mathcal{T}(r) \cdot n(x) \, dS \\
- \frac{\alpha \mu}{8\pi \lambda \eta} \int_S \mathcal{G}(r) \cdot \left[ H' H'(x) - \left( \frac{H'^2(x)}{2} \right) I \right] \cdot n(x) \, dS,
\end{align*}
\]

Again, using the reciprocal identity Eq. (52) for the external flow \( u \) (outside the particle) with a point \( x_0 \) located in the interior of the particle one obtain that

\[
\begin{align*}
- \frac{1}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \sigma(x) \cdot n'(x) \, dS & - \frac{3}{4\pi} \int_S u(x) \cdot \mathcal{T}(r) \cdot n'(x) \, dS \\
+ \frac{\mu}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] \cdot n'(x) \, dS - u^\infty(x_0) = 0.
\end{align*}
\]

The integral representation of the internal flow as a function of the jump condition is obtained by combining (63) and (64). A straightforward algebraic manipulation leads to

\[
\begin{align*}
\lambda u'(x_0) &= u^\infty(x_0) - \frac{1}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \Delta t(x) \, dS - \frac{3\lambda}{4\pi} \int_S [u(x) - \lambda u'(x)] \cdot \mathcal{T}(r) \cdot n(x) \, dS \\
+ \frac{\mu}{8\pi \eta} \int_S \mathcal{G}(r) \cdot \left[ HH(x) - \left( \frac{H^2(x)}{2} \right) I \right] - \alpha \left[ HH'(x) - \left( \frac{H'^2(x)}{2} \right) I \right] \cdot n(x) \, dS.
\end{align*}
\]
5.4. Integral representation for the interface

The integral representation for the flow solution at the interface is found by applying the jump condition \( u(x_0) + \frac{\lambda u'(x_0)}{2} \) to Eqs. (62) and (65). At the limit when \( x_0 \) goes to the interface, the boundary conditions must be applied; \( u(x_0) = u'(x_0) \) (continuity of velocity through the interface), \( H_i = H'_i \) (continuity of tangential component of the magnetic field), \( \mu H_n = \mu H'_n \) (continuity of normal components of the magnetic induction) and the traction discontinuity \( \Delta t \) is given by Eq. (8). Under these conditions only the integral representation for the fluid–fluid interface \( S \) need to be considered, hence

\[
(1 + \lambda)u(x_0) = 2u(x_0) - \frac{1}{4\pi \eta} \int_S \mathcal{G}(r) \cdot \Gamma(\nabla \cdot n) n(x) dS - \frac{3}{2\pi} (1 - \lambda) \int_S u(x) \cdot \mathcal{T}(r) \cdot n(x) dS
\]

\[\]

\[
+ \frac{\mu(1 - \lambda)}{4\pi \eta} \int_S \mathcal{G}(r) \cdot \left\{ \left[ H_i H_i(x) - \left( \frac{H_i^2(x)}{2} \right) 1 \right] - \frac{1}{\alpha} \left[ H_n H_n(x) - \left( \frac{H_n^2(x)}{2} \right) 1 \right] \right\} n(x) dS,
\]

(66)

where the vector field \( H_n = (H \cdot n)n \) and \( H_i = H \cdot (I - nn) \).

5.5. Dimensionless integral representation

All quantities above could be made dimensionless in a context of a three-dimensional ferrofluid drop by using the undisturbed drop size \( a \), the relaxation rate \( \Gamma/\mu a \) and the applied magnetic field intensity \( H_\infty \). In this way, we can define the following dimensionless quantities,

\[
\bar{G}(r) = aG(r), \quad \bar{u} = \eta \Gamma u, \quad \bar{\mathcal{T}}(r) = a^2 \mathcal{T}(r) \quad \text{and} \quad \bar{H} = \frac{H}{H_\infty},
\]

and, we can make Eq. (66) dimensionless. Then, one obtains

\[
\bar{u}(\bar{x}_0) = \frac{2u(\bar{x}_0)}{1 + \lambda} - \frac{1}{4\pi(1 + \lambda)} \int_S (\nabla \cdot n) \bar{G}(\bar{r}) \cdot n(\bar{x}) d\bar{S} - \frac{3(1 - \lambda)}{2\pi(1 + \lambda)} \int_S \bar{u}(\bar{x}) \cdot \bar{\mathcal{T}}(\bar{r}) \cdot n(\bar{x}) d\bar{S}
\]

\[\]

\[
+ \frac{Ca_n(1 - \lambda)}{4\pi(1 + \lambda)} \int_S \bar{G}(\bar{r}) \cdot \left\{ \left[ \bar{H}_i \bar{H}_i(\bar{x}) - \left( \frac{\bar{H}_i^2(\bar{x})}{2} \right) 1 \right] - \frac{1}{\alpha} \left[ \bar{H}_n \bar{H}_n(\bar{x}) - \left( \frac{\bar{H}_n^2(\bar{x})}{2} \right) 1 \right] \right\} n(\bar{x}) d\bar{S},
\]

(68)

where \( \bar{u}(\bar{x}_0) = Ca(\bar{D}^\infty + \bar{W}^\infty) \cdot \bar{x} \). Here, \( \bar{D}^\infty \) and \( \bar{W}^\infty \) denotes the rate of strain and the vorticity tensors of the imposed linear shearing flow respectively, \( Ca = \gamma a\Gamma / \mu \) is the capillary number, that represents the ratio of viscous to surface tension stress, \( Ca_m = \mu \Gamma / \alpha \) is the magnetic capillary number based on the properties of the drop, that represents the ratio of magnetic stresses to surface tension stress. It should be instructive to note that we have opted to call magnetic capillary number the parameter \( Ca_m \) instead of magnetic Bond number (i.e. ratio of body force to interfacial force). We propose this dimensionless parameter since the focus of this work is free surface deformation, and the role of a capillary number is exactly a measure of the relative importance between a force which produces surface stretching to the restoring interfacial force. Therefore this parameter sounds more appropriated to be used in the present context.

In an analogous fashion we can make dimensionless Eq. (26). Since \( \bar{H} \) is irrotational, it implies that \( \bar{H} = \nabla \phi \). Defining the dimensionless potential as being \( \bar{\phi} = \phi / aH_\infty \), one may write

\[
\bar{\phi} = \bar{\phi}(\bar{x}_0) + \left( \frac{1 - \lambda}{\alpha} \right) \int_S \bar{G}(\bar{r}) \nabla \bar{\phi}(\bar{x}) \cdot n(\bar{x}) d\bar{S}.
\]

(69)

6. Final remarks

Eqs. (68) and (69) are considered the key results of the magnetic-hydro-dynamic boundary integral formulation presented here. The analysis described in this paper will be used in a future work to investigate via
numerical simulation the full time-dependent low Reynolds number problem for three-dimensional ferrofluid droplet deformation under the action of shearing motion and magnetic fields, and thereby infer some key properties of magnetic emulsions, when the viscosity ratio and the magnetic permeability ratio of the two phases are not necessarily $\mathcal{O}(1)$. The hydrodynamic integral representation coupled with the magnetic potential integral will determine the drop shape evolution. The mathematical formulation developed here may be extended in a straightforward manner to the problem with multiple polydisperse drops, for the case of general shear flows in the presence of magnetic field where no experimental studies of drop shape evolution are currently available.

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