

Nonlinear Beat Cepheid Models

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Abstract: The numerical hydrodynamic modelling of beat Cepheid behavior has been a longstanding quest in which purely radiative models have failed miserably. We find that beat pulsations occur naturally when *turbulent convection* is included in our hydrodynamics codes. The development of a relaxation code and of a Floquet stability analysis greatly facilitates the search for and analysis of beat Cepheid models. The conditions for the occurrence of beat behavior can be understood with amplitude equations. Here a discriminant \mathcal{D} arises whose sign decides whether single mode or double mode pulsations occur in a model, and this \mathcal{D} depends *only* on the values of the nonlinear coupling coefficients between the fundamental and the first overtone modes. For radiative models \mathcal{D} is always found to be negative, but with sufficiently strong turbulent convection its sign reverses, a necessary condition for double mode pulsations.

The Fourier analysis of the observational data of the *beat Cepheid* light curves and radial velocities shows constant power in two basic frequencies and in their linear combinations which indicates that the stars pulsate in two modes (or more if resonances are involved). Since the beginning of theoretical Cepheid studies in the early 1960s numerical hydrodynamical attempts at modelling the phenomenon beat pulsation have failed, and beat Cepheids have been a black eye in stellar pulsation theory.

In Cepheids energy is carried through the pulsating envelope to the surface by radiation transport as well as by turbulent convection (TC). Even though convection can transport almost all the energy in the hydrogen partial ionization region, this convection is inefficient in the sense that it only mildly affects the structure of the envelope. It was thus generally thought that convection, while important for providing a red edge to the instability strip, would play a minor role the appearance of the nonlinear pulsation. Purely radiative models did indeed give good overall agreement with the observed light and radial velocities. However, recently it has become increasingly clear that there are a number of severe problems with radiative models (Buchler 1998), in addition to their inability to account for beat behavior.

We have recently implemented in our hydrodynamics codes a one dimensional model diffusion equation for turbulent energy (Yecko, Kolláth & Buchler 1998) similar to those advocated by Stellingwerf (1982), Kuhfuss (1986), Gehmeyr & Winkler (1992) and Bono & Stellingwerf (1994). In contrast to these authors, however, we have developed additional tools that allow us to find beat behavior without having to rely on very time-consuming and sometimes inconclusive hydrodynamic integrations

to determine if a model undergoes stable, or steady beat pulsations. These are (a) a linear stability analysis from which we obtain the frequencies and growth rates of *all modes*, (b) a relaxation method (based on the general algorithm of Stellingwerf with the modifications of Kovács & Buchler, 1987) to obtain nonlinear periodic pulsations (limit cycles) when they exist, (c) a stability analysis of the limit cycles that gives their (Floquet) stability exponents.

The 1D turbulent diffusion equation, and the concomitant eddy viscosity, the turbulent pressure and the convective and turbulent fluxes contain (seven) order unity parameters that need to be calibrated through a comparison to observations. In a first paper (Yecko, Kolláth & Buchler 1998) in which we performed a broad survey of the linear properties of TC Cepheid models we found that of these the mixing length, the strengths of the convective flux and of the eddy viscosity play a dominant role and that broad combinations of these three parameters exist that give agreement with the observed widths of both the fundamental and first overtone instability strips. In this Letter we show that the inclusion of TC produces pulsating beat Cepheid models that satisfy the observational constraints, in particular those of period ratios, of pulsation amplitudes and of amplitude ratios. Furthermore the models are very robust with respect to the numerical and physical parameters.

Our discovery of beat Cepheid models has been partially serendipitous. When we started to investigate the nonlinear pulsations of a typical Small Magellanic Cloud Cepheid model ($M = 4.0M_{\odot}$, $L = 1100L_{\odot}$, $X = 0.73$ and $Z = 0.004$ with the the turbulent convective hydrocode we encountered beat pulsations that appeared steady. (We use the 1996 OPAL opacities of Iglesias & Rogers com-

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combined with those of Alexander & Ferguson. The values of the TC parameters – for a definition cf. Yecko et al. (1998) – are $\alpha_c = 3$, $\alpha_\Lambda = 0.41$, $\alpha_p = 0.667$, $\alpha_t = 1$, $\alpha_D = 4$, $\alpha_s = 0.75$, $\alpha_\nu = 1.2$. The steadiness of these beat pulsations was confirmed when several nonlinear hydrodynamics calculations, each initiated with a different admixture of fundamental and first overtone eigenvectors, converged towards the same final steady beat pulsational state. This convergence could be corroborated when we extracted the slowly varying amplitudes with the help of a time-dependent Fourier decomposition, and plotted the resulting phase portraits ($A_0(t)$ vs. $A_1(t)$) that are shown in Fig. 1 where all initializations are seen to converge toward a fixed point located at $A_0 = 0.0104$ and $A_1 = 0.0200$ (These radial displacement amplitudes assume the eigenvectors to be normalized to unity at the stellar surface, $\delta r/r_* = 1$).

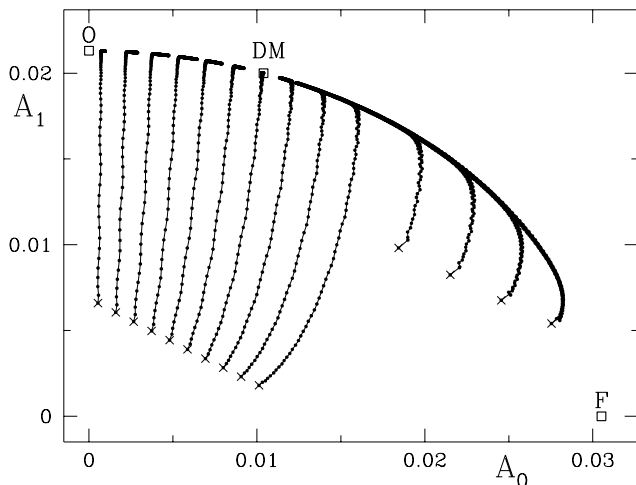


Fig.1: Evolution of the modal amplitudes for different initial conditions (marked with a cross). Equal time intervals between dots. The open squares denote the unstable fundamental (F) and first overtone (O) fixed points and the stable double-mode (DM) point.

While the observed transient behavior of the models provides a conclusive proof of the presence of steady beat pulsations, it is important to explain and describe the behavior on a more fundamental level. The phase portrait of Fig. 1 is very similar to those found for nonresonant mode interaction on the basis of amplitude equations. (Buchler & Kovács 1986, 1987, hereafter BK86 and BK87). We show here that indeed the nonlinear behavior of the hydrodynamical model pulsations can be understood very simply that way.

The amplitudes of the two nonresonantly interacting modes obey remarkably simple equations

$$\frac{dA_0}{dt} = A_0 (\kappa_0 - q_{00}A_0^2 - q_{01}A_1^2) \quad (1a)$$

$$\frac{dA_1}{dt} = A_1 (\kappa_1 - q_{10}A_0^2 - q_{11}A_1^2) \quad (1b)$$

These amplitude equations are 'normal forms' and are

therefore generic for any system in which two modes interact nonresonantly. The assumptions underlying these amplitude equations are satisfied for Cepheids: (a) The lowest modes (fundamental and first overtone here) are weakly nonadiabatic, i.e. the ratios of linear growth rates κ to periods are small, a condition that is readily confirmed by our linear stability analysis; (b) The pulsations are weakly nonlinear which allows a truncation of the amplitude equations in the lowest permissible (third) order; weak nonlinearity can be established by comparing the linear and nonlinear periods which differ less than a tenth of a percent. Furthermore the nonlinear coupling coefficients q_{jk} are always been found to be positive so that amplitude saturation can occur in third order, and it is sufficient to keep terms up to cubic in the amplitudes. (c) In the range of interest there is no important low order resonance of the form $n_0\omega_0 \approx n_1\omega_1 + n_k\omega_k$, with n_0 , n_1 and n_k small positive or negative integers, between the fundamental and the first overtone modes, and possibly a higher mode k if $n_k \neq 0$.

Eqs. (1) have two *single mode* fixed points. The amplitude of the single mode fundamental (0) fixed point is $A_0 = \sqrt{\kappa_0/q_{00}}$ and its linear stability coefficient is $\bar{\kappa}_{1(0)} = \kappa_1 - q_{10}A_0^2$; $\bar{\kappa}_{1(0)}$ measures the stability of the fundamental limit cycle to first overtone perturbations. A positive coefficient implies growth and thus instability. The corresponding first overtone (1) limit cycle amplitude is $A_1 = \sqrt{\kappa_1/q_{11}}$ and its linear stability coefficient is $\bar{\kappa}_{0(1)} = \kappa_0 - q_{01}A_1^2$. The $\bar{\kappa}$'s, when multiplied by the periods P_k of their limit cycles, are equal to the corresponding Floquet exponents (Buchler, Moskalik & Kovács 1991).

Eqs. (1) can also have a *double mode fixed point* whose amplitudes satisfy $A_0^2 = \bar{\kappa}_{0(1)}q_{11}/\mathcal{D}$, $A_1^2 = \bar{\kappa}_{1(0)}q_{00}/\mathcal{D}$, where $\mathcal{D} = q_{00}q_{11} - q_{01}q_{10}$. This fixed point exists provided $A_0^2 > 0$ and $A_1^2 > 0$. If $\mathcal{D} < 0$ then the double mode limit cycle is unstable (when it exists). Stable pulsations occur either in the fundamental *or* first overtone, and the pulsational mode is determined by the evolutionary history of the model (hysteresis). If $\mathcal{D} > 0$ the double mode fixed point is stable, and steady double mode pulsations occur (BK86). Note that this is equivalent to requiring $\bar{\kappa}_{0(1)} > 0$ and $\bar{\kappa}_{1(0)} > 0$, conditions which imply that both *single mode* limit cycles (fundamental and first overtone) are individually unstable. These last two conditions provide the basis for a economical tool to search for double mode behavior, because we can relatively easily compute single mode limit cycles and their stability.

As a further confirmation that the nonresonant scenario applies to the pulsating Cepheid model, we have determined the coefficients of Eqs. (1) as in BK87 by fitting time-dependent solutions of these equations to the temporal variation of the amplitudes in their approach to the limit cycle as shown in Fig. 1. The fitted trajectories in the phase portrait are practically undistinguishable from the hydro results, confirming the applicability and accuracy of the amplitude equation formalism and the

absence of any relevant resonances.

The expression 'double mode Cepheids' is often used cavalierly for beat Cepheids. Since no additional, resonant overtone is involved in the beat pulsations, the latter are thus truly double mode pulsations.

With the relaxation code we are able to compute both the fundamental and the first overtone limit cycles with their respective amplitudes and Floquet stability exponents $\lambda_{1(0)} = P_0 \bar{\kappa}_{1(0)}$ and $\lambda_{0(1)} = P_1 \bar{\kappa}_{0(1)}$. The above discussion then shows that from these four quantities we can extract the four nonlinear q_{jk} coefficients when we have already computed the linear periods and growth rates.

The values we obtain this way for this beat Cepheid model agree quite well with those that we obtain from the fit described in the previous paragraph. Note that these two determinations rely on independent numerical hydrodynamical input, the first on two periodic limit cycles (that are linearly unstable), the second on transient evolution toward the stable double mode pulsation.

In order to investigate the robustness of the observed beat behavior we now explore the pulsational behavior of a *sequence* of Cepheid models in which the effective temperature of the equilibrium modes of the sequence varies from $T_{eff} = 6200$ K to 5800 K. Note that such a sequence is approximately along an evolutionary path. The eddy viscosity parameter α_ν is treated as an additional variable parameter to explore the effect of TC on the behavior.

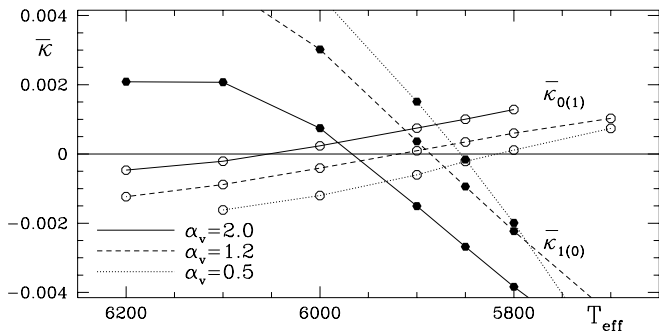


Fig. 2: The limit cycle stability coefficients (d^{-1}) along a Cepheid sequence (i.e. as a function of T_{eff}) for various eddy viscosity strengths α_ν ; fundamental limit cycles $\bar{\kappa}_{1(0)}$ (full circles), first overtone limit cycles $\bar{\kappa}_{0(1)}$ (open circles).

In Fig. 2 the stability coefficients of the sequence are plotted versus T_{eff} , with open/filled circles for those of the fundamental/overtone single mode cycles. The curves are labelled with the corresponding strengths α_ν of the eddy viscosity. As discussed above we expect double mode behavior where both Floquet exponents are positive. (The stability exponents due to perturbations with other modes are always smaller in this sequence and are therefore irrelevant here). For the low value of $\alpha_\nu = 0.5$ (dotted lines) the two stability coefficients are never pos-

itive simultaneously, thus excluding double-mode behavior. On the other hand, for $\alpha_\nu = 1.2$ a double mode region appears between $T_{eff} \sim 5875 - 5915$ K and for $\alpha_\nu = 2.0$ this broadens to $T_{eff} \sim 5965 - 6050$ K.

TABLE 1. Nonlinear Coupling Coefficients (5900 K model).

α_ν	q_{00}	q_{01}	q_{10}	q_{11}	\mathcal{D}	$\bar{\kappa}_{1(0)}$	$\bar{\kappa}_{0(1)}$
0.5	1.20	3.53	4.90	13.31	-1.34	-6.01e-4	1.51e-3
1.0	1.82	3.86	6.64	14.92	1.41	-6.75e-5	6.96e-4
1.2	2.18	4.20	7.55	16.00	3.14	9.17e-5	3.65e-4
1.5	2.89	4.92	9.21	18.17	7.27	3.28e-4	-2.19e-4
2.0	4.68	7.19	13.04	24.23	19.70	7.46e-4	-1.51e-3

How TC does bring about double mode behavior? Fig. 2 shows that, in the region of interest, an increase in the turbulent eddy viscosity causes a rapid decrease in the stability of the fundamental limit cycle ($\bar{\kappa}_{1(0)}$, filled circles), but an increase in that of the first overtone limit cycle ($\bar{\kappa}_{0(1)}$, open circles). However, the slope of $\bar{\kappa}_{0(1)}$ is a lot shallower than that of $\bar{\kappa}_{1(0)}$, and the latter wins out despite its smaller sensitivity to α_ν , thus leading to double mode behavior with increasing eddy viscosity. This description, though, does not tell us whether it is the effect of TC on the linear κ 's or on the nonlinear q 's, or on both, that is responsible for the beat pulsations.

The *necessary* condition for stable double mode pulsations, viz. $\mathcal{D} > 0$, is never found to be satisfied in radiative models. Increasing the strength of the eddy viscosity affects the *nonlinear* coefficients in such a way as to change the sign of \mathcal{D} . Table 1 shows that q_{00} and q_{11} increase faster than q_{01} and q_{10} , making double mode behavior possible for sufficiently large α_ν .

Fig. 3 gives the overall modal selection picture in the $\alpha_\nu - T_{eff}$ plane. The linear edges of the instability region ($\kappa_0 = 0$ and $\kappa_1 = 0$) are shown as dashed lines. By computing the fundamental and first overtone limit cycles for a number of α_ν and T_{eff} values, by interpolation, we can obtain $\bar{\kappa}_{0(1)}$ or $\bar{\kappa}_{1(0)}$ as a function of α_ν and T_{eff} , and in particular the loci where they vanish. The solid curves give the nonlinear pulsation edges and are marked ORE and FBE.

It is straightforward to show that if the two linear growth rates vanish at the same point, the four curves will intersect in a single point on this diagram, that we label *critical point*. (If κ_1 and κ_0 vanish in the same spot, then same is true for the barred κ).

The curve marked OBE is the linear blue edge of the first overtone mode and it coincides with the overtone nonlinear blue edge up to and on the left of the critical point. The linear fundamental blue edge becomes also the fundamental blue edge above the critical point. Above the line ORE $\bar{\kappa}_{0(1)} > 0$ and the first overtone limit cycle is unstable. Below the line FBE $\bar{\kappa}_{1(0)} > 0$ and the fundamental limit cycle is unstable. Thus in the region marked dm both single mode limit cycles are unstable, and this is the region of double-mode pulsation. In the

small triangular region at the bottom, on the other hand, both limit cycles are stable, and either fundamental or first overtone limit cycles can occur.

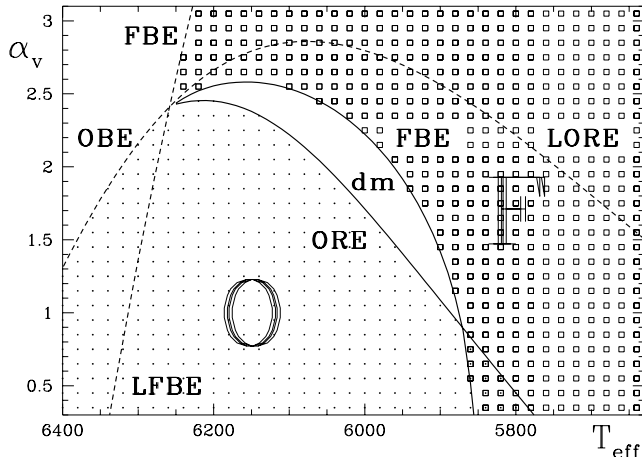


Fig.3: Modal selection in the $\alpha_\nu - T_{eff}$ plane.

In summary, stable first overtone pulsations occur in the dotted region, delineated by the lines OBE and ORE. The fundamental limit cycle is stable in the region marked by open squares, delineated by FBE and FRE (not shown on the far right). This figure makes it particularly evident how TC favors double mode pulsations and why all efforts with radiative codes have failed in modelling beat Cepheids.

We have seen that when TC effects are sufficiently large then the Cepheids should run into the double mode regime in both their crossings of the instability strip. Furthermore, as a Cepheid crosses the double mode regime redward, say, the first overtone amplitude should gradually go to zero while the fundamental amplitude increases from zero to the value it attains as a fundamental mode Cepheid (BK86). The question arises whether this nonresonant scenario is in agreement with the observations.

The four SMC beat Cepheids from the EROS survey (analyzed by Beaulieu and reproduced in Buchler 1998) all have the same amplitude ratios, $A_0/A_1 \sim 0.45$, a priori in disagreement with the nonresonant scenario shown in Fig. 1. of BK86 that suggests that Cepheids with all amplitudes ratios should occur.

In Fig. 4 we display the behavior of the component modal amplitudes of the beat Cepheid models for the $\alpha_\nu = 1.2$ sequence of Fig. 2. The amplitudes of the stable single mode limit cycles are shown as solid lines with solid dots for the fundamental and open dots for the first overtone, and as dashed lines where they are unstable. The fundamental and first overtone component amplitudes of the stable double mode pulsators are shown as solid squares and open diamonds, respectively.

It is seen that although the modal amplitudes do indeed vary continuously throughout as the double mode

regime is traversed, the behavior is very rapid near the cooler side. The reason for this unexpected behavior is that the q 's are *not* constant in this sequence, and what is more, they vary in such a way that \mathcal{D} happens go through zero around 5850 K. It is the presence of this nearby pole causes a change in the curvature of A_0 .

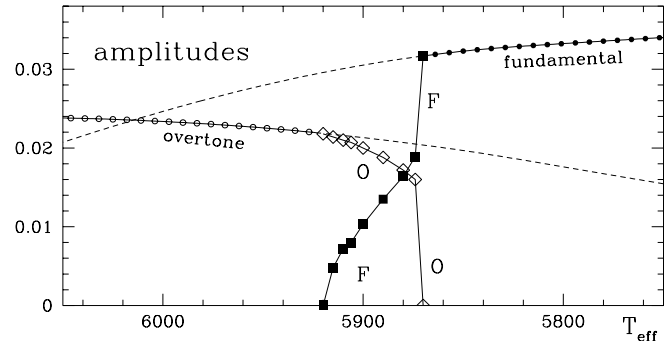


Fig.4: Sequence of models (same as in top of Fig. 2). Amplitudes of the fundamental and of the first overtone limit cycles (lines), solid/dashed where stable/unstable. Double mode component amplitudes of the double mode (fundamental: solid squares and first overtone: open diamonds).

According to Fig. 4 it is therefore much more likely to find beat Cepheids in the slowly varying regime where the ratio $A_0/A_1 \lesssim 0.5$. The computed behavior of the modal amplitudes is thus in agreement with the observed SMC Cepheids, and the nonresonant scenario is consistent with the observations.

We have demonstrated that TC leads naturally to beat behavior in Cepheids, which does not occur with purely radiative models. The reason is that the nonlinear effects of TC dissipation can create a region in which both the fundamental and the first overtone cycles are unstable, and the model undergoes stable double mode pulsations. At a more basic level the amplitude equation formalism shows that it is the effect of turbulent convection on the nonlinear coupling of the fundamental and first overtone modes that allows beat behavior.

The development of a relaxation code (TC) to find periodic pulsations (limit cycles), and a Floquet stability analysis of these limit cycles has made this search quite efficient, and a broader survey of beat Cepheids, with wide ranges of metallicities is in progress. This will also search for beat Cepheid models that pulsate in the first and second overtones.

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