Disturbance growth in primary atomization: three-dimensionality, transient amplification and non-parallel flow effects

Philip Yecko

Biosphere 2 Center, Columbia University, Oracle Arizona 85623, USA

This work examines the growth of disturbances in a two-layer two-phase shear flow with interfacial tension on the boundary between the phases. Instability in such configurations ultimately leads to drop formation and we have been motivated by the need to explain a common three-dimensional feature of primary atomization: the formation of streamwise ligaments. Accordingly, three-dimensional disturbances are studied in a fully viscous temporal framework, based on a matched pair of viscous boundary layer flow profiles. Motivated by recent work on single phase shear flows, we calculate transient amplification of non-modal disturbances; numerical calculations use a Chebyshev collocation method. Optimal disturbances are found to have weak streamwise variability and $\mathcal{O}(1)$ spanwise wavelengths, consistent with ligaments. The maximum growth factors scale with the square of the Reynolds number (as in single phase flow) and also with the viscosity and density ratios. Transient growth properties closely approximate those of single phase boundary layer flow in spite of the dominant presence of the interface and unstable interfacial modes. Mean flow normal to the interface –a non-parallel effect– is also included, but the impact on disturbance growth properties is negligible.

1. Background

In the presence of high velocity gas, a liquid layer undergoes a process of deformation and breakup into droplets. This process, known as airblast atomization, is critical to many applications, including fuel combustion. From observation and experiment a general picture of the initial stages of breakup, or *primary atomization*, has emerged: (1) a laminar two-phase flow emerges from a nozzle; (2) a wave-like instability develops on the interface; (3) wave crests steepen; (4) and are pulled out into liquid sheets; (5) a three-dimensional instability develops; (6) ultimately forming liquid ligaments; and, finally, (7) these ligaments fracture, forming droplets. Atomization ultimately leads to the disintegration of the liquid into a droplet cloud, or spray. Atomization may be thought of as *the transition to turbulence of an interfacial two-phase flow*. Transition is a fundamental problem of fluid mechanics that remains incompletely understood in spite of over a hundred years of study. To unravel the process of transition, it has become conventional to study infinitesimal disturbances; an eigenvalue problem can then be defined by assuming the disturbances have a modal character. Two-dimensional modes in parallel flows are described by the Orr-Sommerfeld (OS) equation; for three-dimensional disturbances, the OS equation is supplemented by the Squire equation. An exponentially growing modal solutions implies instability and was assumed to be a prerequisite for transition.

Boundary layer flows exhibit such a modal instability, in the form of Tollmien-Schlichting (TS) waves, above the critical Reynolds number $\mathrm{Re}_c \approx 520$. Experiments carefully prepared to minimize the influence of free stream turbulence (FST) clearly show the growth of TS modes above Re_c. In other laboratory cases, most notably with high levels of FST, boundary layer flows are found to become turbulent at smaller Reynolds numbers, exhibiting features during transition that do not resemble TS modes. It was therefore believed that some mechanism operating in boundary layer flows allowed turbulence to occur in a way that bypassed the traditional path involving the exponential growth of modes [1]. One such mechanism that has been proposed is transient amplification, a growth process possible in shear flows that was first described by Orr [2] and later elaborated on by Ellingson and Palm [3], Landahl [4] and others [5],[6],[7], [8], [9]; some recent reviews are also available [10],[11], [12].

Transient amplification is a consequence on the fact that the linear operator for disturbances in shear flow is not self adjoint (its eigenfunctions are not orthogonal), making transient growth a generic property of shear flows. Calculation of transient amplification factors can proceed in the form of initial boundary value problem of linear stability theory. A problem's linear operator is used, along with a suitable energy norm, to calculate an amplification factor $G(t)$ of arbitrary infinitesimal initial conditions. In eigenvalue stable problems $G \to 0$ as $t \to \infty$. Still, $G(t)$ can reach relatively large maximum values at intermediate times – large enough, it has been advanced, to induce eigenvalue instability or nonlinear instability in the flow. In single phase flows the kinetic energy norm is sufficient to calculate transient amplification factors, but in two-phase flows this same norm leads to non-convergence [17], [23],[15]. An appropriate norm must account for the movable interface, it is believed. But this issue is not yet completely resolved. In the calculations of Olsson & Henningson [16], for example, the norm included only the energy resulting from surface tension; this was sufficient for numerical convergence. Both Renardy [15] and South & Hooper [23] considered two-phase configurations which had no potential energy (because of equal densities) nor surface tension energy (zero surface tension coefficient). In those cases it was clearly inappropriate to include a potential or surface tension energy in the norm. Still, without some measure $E_h \propto \int |h|^2 dy$ of the interfacial displacement, non-convergence was always encountered.

Another feature of transient growth in shear flows is that the greatest amplification is typically achieved by three-dimensional disturbances (in contrast to eigenvalue instability where Squire's theorem ensures that the most unstable modes are two-dimensional).

The aim of this work is to investigate the role of transient amplification in primary atomization. It is noteworthy that both numerical and laboratory observations of turbulence and transition exhibit features strongly resembling the most amplified transient disturbances (also known optimal perturbations [7]). Streamwise streaks have been linked to flow structures in single-phase shear, such as horseshoe or hairpin vortices, which bear strong resemblance to the ligaments of two-phase flow.

2. Base flow

To describe two-phase flow we use the incompressible Navier-Stokes Equations; to refer to the separate phases we use subscript i where $i = L, G$ to distinguish liquid and gas phases. These equations are:

$$\nabla \cdot \mathbf{u_j} = 0 \quad \text{and} \quad \frac{\partial \mathbf{u_j}}{\partial t} + \mathbf{u_j} \cdot \nabla \mathbf{u_j} = -\frac{\nabla p_j}{\rho_j} + \frac{1}{\mathrm{Re_j}} \nabla^2 \mathbf{u_j} + \frac{1}{\mathrm{We_j}} \kappa \mathbf{n} \delta_I \qquad (1)$$

where δ_I is a distribution on the interface, $\mathbf{u} = u\hat{x} + v\hat{y} + w\hat{z}$, and the notation is otherwise standard. Scaling has been performed using velocity scale U_j^* and length scale δ_j; (dynamic) viscosity μ_j and density ρ_j are assumed constant within each phase. The Reynolds number in each phase is based on the boundary layer thickness and is given as $\mathrm{Re_j} = \rho_j U_j^* \delta_j / \mu_j$; the Weber number in each phase is $\mathrm{We_j} = \rho_j (U_j^*)^2 \delta_j / \sigma$.

The flow examined here is essentially the same as that which we have used previously to study temporal modal instabilities [19]. Quantities in the more dense lower layer fluid are identified by the subscript $i = L$, for liquid, while the in the less dense upper layer fluid subscript $i = G$, denoting gas, is used. The basic state flow is $\mathbf{U} = U_j(y)\hat{x} + V_j^* \hat{y}$ with $U_j(y) = U_j^* \ \mathrm{erf}(y/\delta_{L,G})$. The above U profile is the solution of the *First Stokes Problem* and has characteristic thickness scale δ_j and constant velocity U_j^* far from the interface where $\delta_j = 2\eta^* \sqrt{\mu_j t/\rho_j} = 3.6\sqrt{\mu_j t/\rho_j}$. since from the error function we find $U = 0.99 U^*$ for $\eta^* = 1.8$, There is, in principle, no restriction that $U_G^* > U_L^*$ (as in airblast atomization) and fast liquid into slow ambient gas is also allowed. A transverse component V^* preserves the First Stokes solution as long as $V^* = \text{constant}$; positive V^*, sometimes called blowing, is used here to mimic the expansion of the developing boundary layer and in order to examine the effect of this type of non-parallelism on the properties of disturbance growth.

Velocity is continuous across the interface but the y-derivative of the velocity experiences a discontinuity whose magnitude is proportional to the ratio of the viscosities, as demanded by the continuity of tangential stress: $\tau_L = \mu_L(dU_L/dy) = \tau_G = \mu_G(dU_G/dy)$ at the interface. Error functions are unsteady solutions of the Navier-Stokes equations for parallel flow (see Schlichting & Gersten [22] for details). This base flow develops in time and should be thought of as a snapshot of the flow in which disturbances grow, considering that they may well grow faster than the viscous downstream evolution of the flow profile. This base flow is meant to capture essential elements of a two-fluid mixing layer and is somewhat idealized. Assuming that the base flow evolved directly from an initial step function (or Kelvin-Helmholtz) profile, the boundary layers will satisfy $\delta_G/\delta_L = \sqrt{(\mu_G/\rho_G)t}/\sqrt{(\mu_L/\rho_L)t} = \sqrt{m/r}$, while stress continuity further requires $1/\sqrt{mr} = U_G^*/U_L^*$. In the above we have introduced parameters for the density ratio $r = \rho_G/\rho_L$ and the viscosity ratio $m = \mu_G/\mu_L$.

3. Disturbance Equations

The Navier Stokes equations (1) are perturbed away from the base flow. Following convention, the result is cast as an equation for the velocity component v_j :

$$\left[\left(\frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x} + V_j \frac{\partial}{\partial y} \right) \nabla^2 - U_j'' \frac{\partial}{\partial x} - \frac{1}{\mathrm{Re_j}} \nabla^4 \right] v_j = 0 \qquad (2)$$

and a second equation for the the y vorticity, ζ_j :

$$\left[\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x} + V_j\frac{\partial}{\partial y} - \frac{1}{\mathrm{Re_j}}\nabla^2\right]\zeta_j = -U_j'\frac{\partial v_j}{\partial z} \qquad (3)$$

where the perturbation normal vorticity is: $\zeta_j = \frac{\partial u_j}{\partial z} - \frac{\partial w_j}{\partial x}$, while the prime ($'$) denotes y-differentiation. These equations are supplemented by the boundary conditions: $v_j = \partial v_j/\partial y = \zeta_j = 0$ in the far field, $y \to \pm\infty$.

Because the streamwise and spanwise directions are uniform, perturbations are assumed to have the form: $v_j(x,y,z,t) = \hat{v}_j(y,t)e^{i(\alpha x+\beta z)}$, and $\zeta_j(x,y,z,t) = \hat{\zeta}_j(y,t)e^{i(\alpha x+\beta z)}$.
The resulting equations that govern the behavior of the perturbations $\hat{v}_j, \hat{\zeta}_j$ are:

$$\frac{\partial}{\partial t}(D^2-k^2)\hat{v}_j + (i\alpha U_j + V_jD)\,(D^2-k^2)\hat{v} - i\alpha U_j''\hat{v}_j - \frac{1}{Re_j}(D^2-k^2)^2\hat{v}_j = 0\ , \qquad (4)$$

$$\frac{\partial}{\partial t}\hat{\zeta}_j + (i\alpha U_j + V_jD)\,\hat{\zeta}_j + i\beta U_j'\hat{v}_j - \frac{1}{Re_j}(D^2-k^2)\hat{\zeta}_j = 0 \qquad (5)$$

where $k^2 = \alpha^2+\beta^2$, $D = d/dy$, and the boundary conditions now appear: $\hat{v}_j = D\hat{v}_j = \hat{\zeta}_j = 0$ in the far field.

An eigenvalue problem is formed by assuming $\hat{v}_j(y,t) = \tilde{v}_j(y)e^{-i\omega t}$ and $\hat{\zeta}_j(y,t) = \tilde{\zeta}_j(y)e^{-i\omega t}$ giving one equation for the eigenfunction $\tilde{v}_j$:

$$(-i\omega + i\alpha U_j + V_jD)(D^2-k^2)\tilde{v}_j - i\alpha U_j''\tilde{v}_j - \frac{1}{\mathrm{Re_j}}(D^2-k^2)^2\tilde{v}_j = 0\ , \qquad (6)$$

which is essentially the Orr-Sommerfeld equation but with an additional term in the cross-stream flow V. A second equation is found for the normal vorticity eigenfunction $\tilde{\zeta}_j$:

$$(-i\omega + i\alpha U_j + V_jD)\tilde{\zeta}_j + i\beta U_j'\tilde{v}_j - \frac{1}{\mathrm{Re_j}}(D^2-k^2)\tilde{\zeta}_j = 0\ , \qquad (7)$$

which is known as the Squire equation (again with an additional term in V). The boundary conditions in the far field now appear: $\tilde{v}_j = D\tilde{v}_j = \tilde{\zeta}_j = 0$; from now on, the tilde ($\sim$) symbols will be dropped for clarity.

The conditions that must hold at the interface of two viscous fluids are the continuities of the tangential velocity, u: $\omega(v_G'-v_L') = \alpha(U_L'v_L - U_G'v_G)$ and of the normal velocity, v: $v_G = v_L$, both evaluated on $y = 0$. The continuity of the spanwise tangential velocity component, w is guaranteed by the continuity of u and of η (below). The two tangential stresses must also be continuous at $y = 0$, giving: $m(v_G'' + \alpha^2 v_G) = v_L'' + \alpha^2 v_L$ and:

$$m(\beta v_G'' + \beta k^2 v_G + \alpha\eta_G') = \beta v_L'' + \beta k^2 v_L + \alpha\eta_L'\ . \qquad (8)$$

The normal stress condition on the interface is given by:

$$r(\omega v_G' + \alpha U_G' v_G) - (\omega v_L' + \alpha U_L' v_L) + \frac{m(v_G''' - 3k^2 v_G')}{i\mathrm{Re}} - \frac{(v_L''' - 3k^2 v_L')}{i\mathrm{Re}} = \frac{\alpha k^2}{\mathrm{We}}\frac{v_L' - v_G'}{U_L' - U_G'}\ . \qquad (9)$$

The normal vorticity is also subject to a matching condition at the interface, of the form: $\omega(\zeta_L - \zeta_G) = \beta(U'_G v_G - U'_L v_L)$.

The above conditions are applied at the perturbed interface, whose position is described by the quantity $y = he^{i(\alpha x+\beta z-\omega t)}$, where the kinematic condition gives $h = v(0)/(i\alpha U + i\beta V - i\omega)$. The interface displacement h is an essential component of the total "eigenfunction" of the two-phase flow, supplementing the normal velocities v_j and normal vorticities ζ_j in each layer.

To simplify the above interface conditions, the Reynolds and Weber numbers were assumed to be based on the liquid layer quantities. The resulting Weber number that appears above is then the liquid Weber number:

$$\mathrm{We} = \mathrm{We_L} = \frac{\rho_L (U_L^*)^2 \delta_L}{\sigma}, \tag{10}$$

even though it is more customary when treating strong density ratios to consider the gas, or aerodynamic, Weber number; the two Weber numbers are simply related as follows: $\mathrm{We_G} = \mathrm{We_L}/rm$.

In calculating temporal stability of disturbances, α, β are assumed to be purely real while ω is complex; a mode having $\mathrm{Im}(\omega) < 0$ will be temporally unstable. Instability in strongly developing flows –such as atomizing interfaces– is often described using spatial theory since unstable modes are known to grow as they propagate downstream. Spatial and temporal results can be simply related using Gaster's transformation in the neighborhood of neutral stability. Transient growth may also be approached temporally or spatially. For single phase boundary layer flows, optimal disturbances found from spatial theory are nearly identical to those found using temporal theory (see, eg., [7],[14],[13]), the primary difference being that the maximum $G(t) \sim \mathcal{O}(\mathrm{Re})$ in spatial theory, while the maximum $G(t) \sim \mathcal{O}(\mathrm{Re}^2)$ in temporal theory.

4. Transient Amplification

An initial boundary value problem (IBVP) is formed, closely following the method of Schmid and Henningson [12], by defining initial conditions $v_j(0) = \zeta_j(0) = h(0) = 0$. It is convenient to also define $\hat{\mathbf{q}}_\mathbf{j} = \mathbf{q}_j e^{-i\omega t}$ where $\mathbf{q}_\mathbf{j} = (v_G, \zeta_G, h, v_L, \zeta_L)^T$. Then the IBVP can be written $\mathbf{M}\partial_t \hat{\mathbf{q}}_j + \mathbf{L}\hat{\mathbf{q}}_j = 0$ where

$$\mathbf{M} = \begin{pmatrix} D^2-k^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D^2-k^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;,\; \mathbf{L} = \begin{pmatrix} \mathcal{OS}_G & 0 & 0 & 0 \\ \mathcal{R}_G & \mathcal{SQ}_G & 0 & 0 \\ 0 & 0 & \mathcal{OS}_L & 0 \\ 0 & 0 & \mathcal{R}_L & \mathcal{SQ}_L \end{pmatrix}$$

and the matrix elements are:

$$\mathcal{OS}_j = (i\alpha U_j + V_j D)(D^2 - k^2) - i\alpha U_j'' - \frac{1}{\mathrm{Re_j}}(D^2 - k^2)^2, \tag{11}$$

$$\mathcal{SQ}_j = i\alpha U_j + V_j D - \frac{1}{\mathrm{Re_j}}(D^2 - k^2) \;, \quad \text{and} \quad \mathcal{R}_j = i\beta U_j' \tag{12}$$

Using the above, the eigenvalue problem can be rewritten: $\mathbf{L}\tilde{\mathbf{q}}_j = i\omega\mathbf{M}\tilde{\mathbf{q}}_j$ or $i\omega\tilde{\mathbf{q}}_j = \mathbf{M}^{-1}\mathbf{L}\tilde{\mathbf{q}}_j = \mathcal{N}\tilde{\mathbf{q}}_j$, defining the matrix operator $\mathcal{N}$. The disturbance problem in this form is used below to calculate transient amplification; first we describe the numerical technique.

A Chebyshev collocation code developed in previous work [19] was used to evaluate the stability problem defined above. Transient amplification calculations were implemented according to the method given by Reddy and Henningson [9]. To facilitate the expansion, the linear problem is mapped to the interval $[-1, 1]$ on which the Chebyshev polynomials are orthogonal. The eigenfunctions can then be written as an expansion in a finite number N of Chebyshev terms $T_n(y)$ with unknown expansion coefficients, where N is the number of polynomials used, typically 100 or less. Derivatives of eigenfunctions are evaluated using the recurrence relations based on linear combinations of lower order Chebyshev polynomials. Because of the infinite domain a continuous spectrum is present in addition to the finite discrete spectrum. In the calculations performed here, the infinite domain was approximated by a sufficiently large domain, a method known to be effective [7]. The eigenvalue problem is transformed into a corresponding $2N \times 2N$ matrix problem for the expansion coefficients including the boundary and matching conditions.

Following the formulation of transient growth used in [12], let $G(t)$ represent the maximum possible energy amplification at time t, where G is optimized over all possible initial conditions for each instant in time. Then define

$$G(t) = \max\frac{||\mathbf{q(t)}||^2}{||\mathbf{q_0}||^2} = ||e^{i\mathcal{N}t}||^2 = ||e^{-it\Lambda}||^2 \ ,$$

where $\mathbf{q_0}$ is an initial disturbance, $\mathcal{N}$ is the linear operator defined above, and Λ is the diagonal matrix of eigenvalues: $\{\omega_1, \omega_2, ...\}$. The matrix exponential is easily evaluated using a series expansion of e^x.

Critical to the accurate calculation of $G(t)$ is the use of an energy norm $||\cdot||$ appropriate for the problem. Originally, Renardy [15] defined a norm (the "h-norm") which accounted for the displaced interface via a term of the form h^*h, where the * denotes complex conjugate. Even in the absence of interfacial potential energy terms (when there is no gravity and no surface tension for example) it was found that an interface term of this kind was needed in the norm to ensure numerical convergence. South and Hooper [23] later verified this, using the h-norm, for a problem with no interface potential energy. On the other hand, when a potential energy is present, the inclusion of the relevant potential term (which is typically of the form h^*h) in the energy norm has also been found to ensure convergence [16], [18]. Since we consider non-zero surface tension here, an energy norm of the following form is adopted:

$$||q|| = \frac{1}{2k^2}[\int_{-L}^{0}(|Dv_L|^2 + k^2|v_L|^2 + |\zeta_L|^2)dy + \int_{0}^{L}(|Dv_G|^2 + k^2|v_G|^2 + |\zeta_G|^2)dy + \frac{k^4}{\mathrm{We}}|h|^2].$$

5. Results

Two-phase flow consisting of matched error function profiles exhibits strongly amplified disturbances of truly three-dimensional character in the parameter range characteristic of atomization of water by air. In this preliminary work the density ratio is fixed at either $r = 0.1$ or

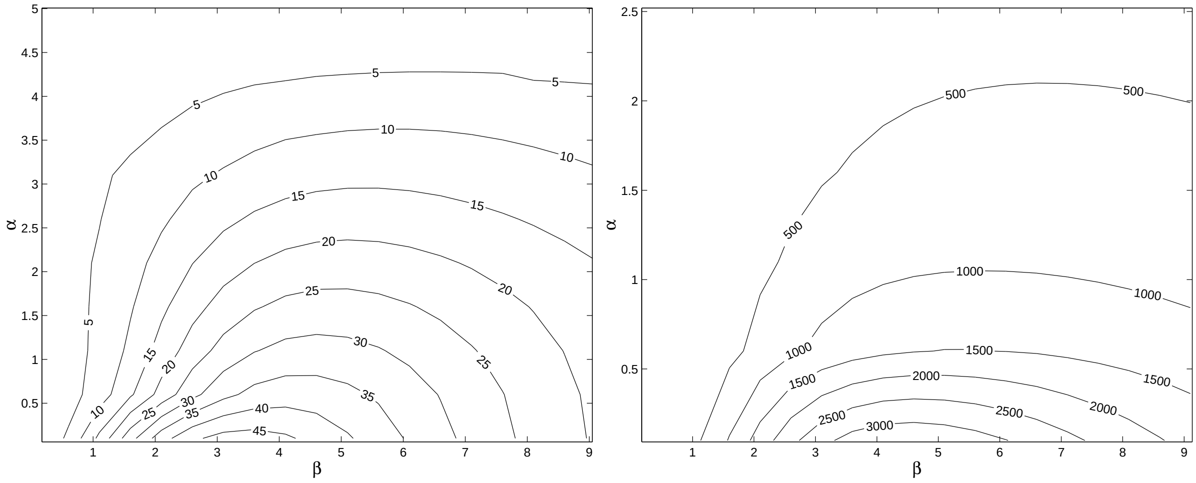

FIG. 1: Maximum amplification factors G_{MAX} of three-dimensional disturbances in two-phase shear flow with $m = 10^{-1}$, $\mathrm{Re} = 100$ and $\mathrm{We} = 100$ and (a) $r = 10^{-1}$ as compared to (b) $r = 10^{-3}$

$r = 0.001$, the viscosity ratio at $m = 0.1$, the Weber number $\mathrm{We} = 100$ and the Reynolds number $\mathrm{Re} = 100$. We are faced with a vast parameter space to explore, so it is necessary to restrict attention to specific ranges of interest.

In looking at the largest value G_{MAX} achieved by the function $G(t)$, it is possible to identify (see Fig.1a) a largest or *peak* $G_{\mathrm{MAX}} \sim 50$ falling in the vicinity $\alpha \approx 0$ and $\beta \approx 4$. Such a disturbance has weak streamwise variation but a strong spanwise variation, consistent with the nature of a three-dimensional disturbance needed to initiate ligament growth. For stronger density contrasts, the peak amplification factors increase dramatically. In Fig.1b the peak $G_{\mathrm{MAX}} \sim 3500$, but still occurs near $\beta \approx 4$ and $\alpha \approx 0$. Note that eigenvalue instability, although present, is not plotted here. Exponential modal growth leads, naturally, to diverging values of G, since growth is not transient. The values plotted in the figure have therefore been computed by neglecting the eigenvalue unstable mode from the computation of G.

6. Conclusions

This study has been motivated by the ubiquity of three-dimensional ligament-like structures in a variety of interface breakup settings (see Lefebvre [24], Lasheras and Hopfinger[21], Lin and Reitz [20], Liu [25]). We have concentrated on three-dimensional disturbances in two-phase sheared layers and the characteristics of their transient growth. A formalism including a transverse non-parallel velocity component is presented. Preliminary tests have showed no significant difference in the properties of disturbances when this non-parallel effect is included. We have found that transient amplification is not only possible but exhibits large amplification factors, especially for strong density contrasts. The structure of the peak amplification factors in wavenumber space strongly resembles that of single-phase boundary layer flows, implying that the flow profile controls the transient growth.

7. References

[1] MORKOVIN, M.V. “The many faces of transition,” in C. S. Wells, ed., *Viscous Drag Reduction*. Plenum Press, (1969).

[2] ORR, W.M.F. “The stability or the instability of the steady motions of a perfect liquid and of a viscous liquid,” Proc. R. Irish Acad. A **27**, p.9-138. (1907).
[3] ELLINGSON, T. AND PALM, E. “Stability of linear flow,” Phys. Fluids **18(4)**, p.487. (1975).
[4] LANDAHL, M.T. “A note on an algebraic instability of inviscid parallel shear flows,” J. Fluid Mech. **98**, p.243. (1980).
[5] HULTGREN, L.S. AND GUSTAVSSON, L.H. “Algebraic growth of disturbances in a laminar boundary layer,” Phys. Fluids **24**, p.1000. (1981).
[6] BOBERG, L. AND BROSA, U. “Onset of turbulence in a pipe,” Z. Naturforschung **43a**, p.697. (1988).
[7] BUTLER, K.M. AND FARRELL, B.F. “Three-dimensional optimal perturbations in viscous shear flow,” Phys. Fluids A **4**(8), p1637-1650 (1992).
[8] TREFETHEN, TREFETHEN, REDDY AND DRISCOLL “Hydrodynamic instability without eigenvalues,” Science **261**, 578-584 (1993).
[9] REDDY SC, HENNINGSON DS “Energy growth in viscous channel flows,” J. Fluid Mech. **252**, p209-238 (1993).
[10] GROSSMANN, S. “The onset of shear flow turbulence,” Rev. Mod. Physics **72**(2), p603-618 (2000).
[11] RESHOTKO, E. “Transient growth: A factor in bypass transition,” Phys. Fluids **13(5)**, p1067 (2001).
[12] P.J. SCHMID AND D.S. HENNINSON *Shear Flow Instability* (Springer-Verlag, New York, 2000)
[13] CORBETT, P. AND BOTTARO, A. “Optimal perturbations for boundary layers subject to streamwise pressure gradient,” Phys. Fluids **12(1)**, p.120. (2000).
[14] BREUER, K.S. AND KURAISHI, T. “Transient growth in two- and three-dimensional boundary layers,” Phys. Fluids **6(6)**, p.1983. (1984).
[15] RENARDY, Y. “The thin-layer effect and interfacial stability in a two-layer Couette flow with similar liquids,” Phys. Fluids **30(6)**, p.1627. (1987).
[16] OLSSON, P.J. AND HENNINGSON, D.S. “Optimal disturbance growth in watertable flow,” Stud.Appl.Math **94** p.183. (1995).
[17] VAN NOORDER, T.L., BOOMKAMP, P.A.M., KNAAP, M.C. AND VERHEGGEN, T.M.M. “Transient growth in parallel two-phase flow: Analogies and differences with single-phase flow,” Phys. Fluids **10(8)**, p.2099. (1998).
[18] DE LUCA, L., COSTA, M. AND CARAMIELLO, C. “Energy growth of initial perturbations in two-dimensional gravitational jets,” Phys. Fluids **14(1)**, p.289. (2002).
[19] YECKO, P., ZALESKI, S. AND FULLANA, J.-M. “Viscous modes in two-phase mixing layers,” Physics Fluids **14(12)**, p4115-4122 (2002).
[20] S.P. LIN AND R.D. REITZ “Drop and spray formation from a liquid jet,” Ann. Rev. Fluid Mech. **30**, 85 (1998).
[21] J.C. LASHERAS AND E.J. HOPFINGER “Liquid jet instability and atomization in a coaxial gas stream,” Ann. Rev. Fluid Mech. **32**, 275 (2000).
[22] SCHLICHTING, H. AND GERSTEN, K. Boundary Layer Theory, 8th edition, Springer, Berlin (2000).
[23] M.J. SOUTH AND A.P. HOOPER “Linear growth in two-fluid plane Poiseuille flow,” J. Fluid Mech. **381**, 12 (1999).
[24] A.H. LEFEBVRE *Atomization and Sprays* (Taylor and Francis, New York, 1990)
[25] LIU, H. *Science and Engineering of Droplets* (Noyes Publications, New York, 2000)