A LEMMA ON EVOLUTION OPERATORS AND APPLICATIONS

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Abstract. In this note we characterize a family of evolution operators that can be applied to prove the existence and the uniqueness of the solutions of a general class of non-autonomous partial differential equations. In terms of applications we consider a system of parabolic equations with a non-diagonal time dependent diffusion matrix, the wave equation with time dependent damping and a thermoelastic plate equation with time dependent coefficients.

1. Introduction

In [4], Leiva shows that if \( \{A_n\}_{n \geq 0} \) and \( \{P_n\}_{n \geq 0} \) are two families of bounded linear operators on a Hilbert Space, \( Z \), where \( \{P_n\}_{n \geq 0} \) is a complete orthogonal sequence of projections, then under the certain conditions, \( \{T(t)\}_{t \geq 0} \), given by

\[
T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z, \quad t \geq 0,
\]

is a \( C_0 \)-semigroup. Furthermore, the infinitesimal generator

\[
A : D(A) \to Z
\]

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of \{T(t)\}_{t \geq 0} is,

\[\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \; z \in D(\mathcal{A}),\]

where the domain of \(\mathcal{A}\) is

\[D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n\|^2 < \infty\}.\]

The paper also completely characterizes the spectrum of \(\mathcal{A}, \sigma(\mathcal{A})\), in terms of the spectra of the operators \(A_n, n \geq 1\).

This research extends the work done in [4] to evolution operators. Namely, we are interested in the abstract Cauchy problem defined on a Hilbert Space \(Z\),

\[
\left\{ \begin{array}{l}
\frac{dz}{dt} = \mathcal{A}(t)z(t), \; 0 \leq tT \\
z(0) = z_0 \in Z
\end{array} \right.
\]

where \(\mathcal{A}(t)\) is a family of unbounded linear operators from \(D\) to \(Z\), strongly continuous in \(t\). The domain of \(\mathcal{A}(t), D\), is constant and dense in \(Z\). We motivate our analysis through the following initial boundary value problem of a system of parabolic equations with a time dependent **non-diagonal diffusion coefficient matrix**.
Example 1.1. Consider the following initial boundary value problem defined on a smooth bounded domain, $\Omega \subset \mathbb{R}^n$.

$$
\begin{align*}
&\begin{cases}
    u_t = a(t) \triangle u + b(t) \triangle v \\
    v_t = c(t) \triangle u + d(t) \triangle v \quad t > 0
\end{cases} \\
&u(0, x) = \psi_1(x), \; v(0, x) = \psi_2(x) \; x \in \Omega \\
&u = v = 0 \text{ on } \partial \Omega
\end{align*}
$$

where the functions, $a(t), b(t), c(t), d(t) > 0$ for all $t \geq 0$ are continuous.

Let $H = L_2(\Omega)$ and define $A : D \to H$ to be the linear unbounded operator:

$$
A\phi = -\triangle \phi,
$$

where the domain of $A$ is $D = H^1_0(\Omega) \cap H^2(\Omega)$. The spectral properties of $A$ are well known and $A$ can be decomposed as:

$$
Ay = \sum_{j=1}^{\infty} \lambda_j E_j y,
$$

where $\{-\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the Laplacian and $\{E_j\}$ is the complete orthogonal sequence of projections,

$$
E_j x = \sum_{k=1}^{\gamma_j} \langle \phi_{j,k}, x \rangle \phi_{j,k},
$$

defined through the eigenvectors $\{\phi_{j,k}\}_{k=1}^{\gamma_j}$ corresponding to eigenvalue $\lambda_j$ for each $j$.

Utilizing this decomposition, System (1) can be written abstractly as
\[
\begin{cases}
    z' = A(t)z, \ t > 0 \\
    z(0) = \psi \in Z = H \times H
\end{cases}
\]

The vector, \( z = (u, v)^T \) and \( A(t) \) is the unbounded linear operator

\[
A(t) = \begin{pmatrix}
    -a(t)A & -b(t)A \\
    -c(t)A & -d(t)A
\end{pmatrix}
\]

with common domain \( D(A(t)) = D(A) \times D(A) \).

If \( \{P_n\}_{n=1}^\infty \) is the sequence of orthogonal projections on \( Z \) defined by

\[
P_n = \begin{pmatrix}
    E_n & 0 \\
    0 & E_n
\end{pmatrix},
\]

and \( \{A_n(t)\}_{n=1}^\infty \) is the sequence of bounded linear operators

\[
A_n(t) = -\lambda_n \begin{pmatrix}
    a(t) & b(t) \\
    c(t) & d(t)
\end{pmatrix} P_n
\]

then we may decompose \( A(t) \) by

\[
A(t)z = \sum_{n=1}^\infty A_n(t)P_nz,
\]

for each \( z \in D(A(t)) \).

The spectral decomposition of \( A(t) \) yields the following important observations.
1. Each \( A_n(t) \) commutes with its corresponding projection,

\[
A_n(t)P_n = P_nA_n(t),
\]

for each \( n \geq 1 \).

2. Each \( A_n(t) \) uniformly continuous in \( L(Z) \).

3. Each \( A_n(t) \) generates an evolution operator \( U_n(t, s) \), \( 0 \leq s \leq t \) such that

\[
\frac{\partial}{\partial t}U_n(t, s) = A_n(t)U_n(t, s) \quad \text{for} \quad 0 \leq s \leq t
\]

\[
\frac{\partial}{\partial s}U_n(t, s) = -U_n(t, s)A_n(s) \quad \text{for} \quad 0 \leq s < t
\]

\[
\|U_n(t, s)\| \leq g(t, s), \quad t \leq s
\]

4. The evolution operator generated by \( A(t) \) is given by

\[
U(t, s)z = \sum_{n=1}^{\infty} U_n(t, s)P_nz, \quad \text{for} \quad z \in Z.
\]

5. The spectrum of \( A(t) \), \( \sigma(A(t)) \), is given by

\[
\bigcup \sigma(A_n(t))
\]

where \( \tilde{A}_n(t) = A_n(t)P_n \).

Motivated by the properties of Example 1.1, we extend the notions and concepts in Example 1.1 to a large class of non-autonomous partial differential equations. Mainly
we are interested if a densely defined unbounded linear operator, $A(t)$ on a constant
domain $D$ has a decomposition

$$A(t)z = \sum_{n=1}^{\infty} A_n(t)P_n z \quad \text{for } z \in D,$$

where $A_n(t)$ are bounded linear operators and $P_n$ are projections for each $n \geq 1$, how
can we extract the spectral properties of the simpler bounded operators, $A_n(t)$, to
characterize the spectrum of the more complex unbounded operator, $A(t)$.

The next section formalizes the general setting of our research.

2. Evolution Operators

We begin this section with the definition of evolution operators and the generator
of such operators.

**Definition 2.1.** A two parameter family of bounded linear operators, $U(t, s)$, $0 \leq s \leq t \leq T$ on $Z$ is called an evolution operator if the following two conditions are satisfied:

(i) $U(s, s) = I$ and $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq t \leq T$.

(ii) $(t, s) \rightarrow U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Now, we shall give a weaker definition of generator of an evolution operator that is
convenient for our purpose.
**Definition 2.2.** (The Generator of an Evolution Operator) The generator $\mathcal{A}(t)$ of an evolution operator $U(t, s)$, $0 \leq s \leq t \leq T$ is defined as follows:

$$\mathcal{A}(t) = \lim_{h \to 0^+} \frac{U(t + h, t)z - z}{h}, \quad z \in D, \quad (3)$$

where

$$D = \left\{ z \in Z : \lim_{h \to 0^+} \frac{U(t + h, t)z - z}{h} \text{ exists, } 0 \leq t \leq T \right\}. \quad (4)$$

**Lemma 2.1.** Let $U(t, s)$, $0 \leq s \leq t \leq T$ be an evolution operator on $Z$ such that $U(t, s)z \in D$, for all $z \in D$ and $0 \leq s \leq t \leq T$. Assume there exists a real valued continuous nonnegative function $g(t, s)$ with $\|U(t, s)\| \leq g(t, s)$ for all $0 \leq s \leq t \leq T$. Then for all $z \in D$ we have that

$$\frac{\partial}{\partial t} U(t, s)z = \mathcal{A}(t)U(t, s)z \quad \text{for } 0 \leq s \leq t \quad (5)$$

$$\frac{\partial}{\partial s} U(t, s)z = -U(t, s)\mathcal{A}(s)z \quad \text{for } 0 \leq s < t \quad (6)$$

**Proof.** If $z \in D$, for $0 \leq s \leq t \leq T$,

$$\lim_{h \to 0^+} \frac{U(t + h, s)z - U(t, s)z}{h} = \lim_{h \to 0^+} \frac{U(t + h, t)U(t, s)z - U(t, s)z}{h}. \quad (7)$$

Since $U(t, s)z \in D$ we obtain that

$$\lim_{h \to 0^+} \frac{U(t + h, s)z - U(t, s)z}{h} = \mathcal{A}(t)U(t, s)z.$$
Now, suppose $t > 0$, $t > s$ and $h > 0$ is small enough such that $t - h > 0$ and $t - h > s$.

Then

$$\lim_{h \to 0^+} \frac{U(t - h, s)z - U(t, s)z}{-h} = \lim_{h \to 0^+} \frac{U(t - h)U(t - h, s)z - U(t - h, s)z}{h}.$$

Making the change of variable $u = t - h$ and observing that $u$ goes to $t$ as $h$ goes to zero, we obtain

$$\lim_{h \to 0^+} \frac{U(t - h, s)z - U(t, s)z}{-h} = \lim_{h \to 0^+} \frac{U(u + h, u)U(u, s)z - U(u, s)z}{h} = \mathcal{A}(t)U(t, s)z.$$

If $s = t$, the definition of $\mathcal{A}(t)$ yields

$$\mathcal{A}(t)z = \mathcal{A}(t)U(t, t)z = \lim_{h \to 0^+} \frac{U(t + h, t)z - U(t, t)z}{h}.$$

Hence

$$\frac{\partial}{\partial t} U(t, s)z = \mathcal{A}(t)U(t, s)z \quad \text{for} \quad 0 \leq s \leq t.$$

Now, suppose $t > s$ and $h > 0$ is small enough such that $s + h < t$. Then

$$\left\| \frac{U(t, s + h)z - U(t, s)z}{h} + U(t, s)A(s)z \right\| = \left\| \frac{U(t, s + h)z - U(t, s + h)U(s + h, s)z}{h} + U(t, s + h)U(s + h, s)A(s)z \right\| = \left\| -U(t, s + h) \left\{ \frac{U(s + h, s)z - z}{h} - U(s + h, s)A(s)z \right\} \right\| \leq g(t, s) \left\| \frac{U(s + h, s)z - z}{h} - U(s + h, s)A(s)z \right\| - g(t, s)\|A(s)z - A(s)z\| = 0, \quad \text{as} \quad h \to 0^+.$$
The proof of the case where \( h \to 0^- \) is analogous.

Therefore,

\[
\frac{\partial}{\partial s} U(t, s)z = -U(t, s)A(s)z \quad \text{for} \quad 0 \leq s < t.
\]

\[\square\]

**Lemma 2.2.** Let \( t > 0 \). Under the conditions of Lemma 2.1 \( A(t) : D \to Z \) is a closed linear operator.

**Proof.** Let \( \{z_k\} \) a sequence in \( D \) and \( y \in Z \) such that \( z_k \to z \) and \( A(t)z_k \to y \). Then

\[
U(t+h,t)z_k - U(t+h,t+h)z_k = -\int_t^{t+h} \frac{\partial U(t,s)z_k}{\partial s} ds
- \int_t^{t+h} U(t+h,s)A(s)z_k ds.
\]

Then,

\[
\frac{U(t+h,t)z_k - z_k}{h} = \frac{1}{h} \int_t^{t+h} U(t+h,s)A(s)z_k ds.
\]

Passing to the limit as \( k \to \infty \) and \( h \to 0^+ \) we obtain that

\[
\lim_{h \to 0^+} \frac{U(t+h,t)z - z}{h} = U(t,t)y = y.
\]

Therefore, \( z \in D \) and \( A(t)z = y \). \[\square\]

**Theorem 2.3.** Let \( U(t,s) \), \( 0 \leq s \leq t \leq T \) be an evolution operator on \( Z \) satisfying the conditions on Lemma 2.1, and \( A(t) \) its generator with domain \( D \). Then the
Cauchy problem

\[
\begin{cases}
  z'(t) = A(t)z(t), & t > 0 \\
  z(s) = z_0, & z_0 \in D, \quad 0 \leq s < t,
\end{cases}
\]

has the unique solution

\[
z(t) = U(t,s)z_0, \quad t \geq s.
\]

Proof. From Lemma 2.1 we get that \( z(t) = U(t,s)z_0 \) is one solution of the Cauchy problem. Now, we shall prove the uniqueness, to this end we will suppose that \( y(t) \) is another solution of the problem, and consider the following function:

\[
F(u) = U(t,u)y(u), \quad 0 \leq u \leq t.
\]

Then

\[
\dot{F}(u) = \frac{\partial U(t,u)y(u)}{\partial u} + U(t,u)\dot{y}(u) = -U(t,u)A(u)y(u) + U(t,u)A(u)y(u) = 0.
\]

Therefore, \( F(u) = U(t,u)y(u) = \text{constant} \). Hence,

\[
F(s) = F(t) \iff U(t,s)y(s) = U(t,t)y(t) \iff y(t) = U(t,s)z_0.
\]

We omit the proof of the following Theorem since it is analogous to the strongly continuous semigroup case (see for example [2], [7]).
Theorem 2.4. Let $U(t,s)$, $0 \leq s \leq t \leq T$ be an evolution operator on $Z$ satisfying the conditions on Lemma 2.1, and $A(t)$ its generator with domain $D$. Consider the non-homogeneous Cauchy problem

$$
\begin{cases}
  z'(t) = A(t)z(t) + f(t), & t > 0 \\
  z(s) = z_0, & z_0 \in Z, \ s \geq 0.
\end{cases}
$$

(9)

Assume either

i) $z_0 \in D$ and $f \in C(\mathbb{R}_+, Z)$ takes values in $D$ and $A(\cdot)f(\cdot) \in C(\mathbb{R}_+, Z)$, or

ii) $z_0 \in D$ and $f \in C^1(\mathbb{R}_+, Z)$.

Then (9) has a unique solution $z \in C^1(\mathbb{R}_+, Z)$ with value in $D$.

Moreover, this solution $z(t)$ is a solution of the following integral equation

$$z(t) = U(t, 0)z_0 + \int_0^t U(t, s)f(s)ds.$$ 

(10)

Definition 2.3. Note that any solution $z$ of the problem (9) satisfies the integral equation (10), but not conversely since a solution of this integral equation is not necessarily differentiable. We shall called continuous solutions of (10 a mild solution of (9).

3. Main Theorem

In this section we shall prove a lemma that characterizes a family of evolution operators that can be applied to prove the existence and the uniqueness of the solutions of a general class of non-autonomous partial differential equations.
Lemma 3.1. Let \( \{P_n\}_{n \geq 1} \) be a sequence of complete orthogonal projections on \( Z \) such that for each \( z \in Z \) and \( \{A_n(t)\}_{n=1}^{\infty}, t \in [0, T] \) be a sequence of bounded linear operators continuous in \( t \) in the uniform operator topology with corresponding evolution operator \( U_n(t, s) \) such that

\[
A_n(t)P_n(t) = P_n(t)A_n(t), \quad n \geq 1, \quad 0 \leq t \leq T.
\]

Define the two parameter family of linear operators for \( z \in Z \) by

\[
U(t, s)z = \sum_{n=1}^{\infty} U_n(t, s)P_nz. \tag{11}
\]

Then, the following statements holds:

1. If there exists a real valued continuous nonnegative function \( g(t, s) \) such that

\[
U_n(t, s) \leq g(t, s), \quad 0 \leq s \leq t \leq T, \quad n \geq 1,
\]

then \( \{U(t, s)\}_{0 \leq s \leq t \leq T} \) is an evolution operator.

2. The generator \( A(t) : D \to Z \) of \( \{U(t, s)\}_{0 \leq s \leq t \leq T} \) is given by

\[
A(t)z = \sum_{n=1}^{\infty} A_n(t)P_nz, \quad z \in D,
\]

where

\[
D = \{ z \in Z : \sum_{n=1}^{\infty} \|A_n(t)P_nz\|^2 < \infty, \quad \forall t \in [0, T] \}.
\]

3. The spectrum of \( A(t) \) is given by

\[
\sigma(A(t)) = \bigcup_{n=1}^{\infty} \sigma(A_n(t))
\]
where $\bar{A}_n(t) = A_n(t)P_n$.

Proof. We first show that $U(t, s)$ is a bounded linear operator for fixed $s, t$. In fact, let $z \in Z$. Then

$$
\|U(t, s)z\|^2 = \left\| \sum_{n=1}^{\infty} U_n(t, s)P_nz \right\|^2 \\
\leq g^2(t, s) \sum_{n=1}^{\infty} \|P_nz\|^2 \\
= g^2(t, s)\|z\|^2.
$$

Therefore $\|U(t, s)x\| \leq g(t, s)\|z\|$. This proves that $U(t, s)$ is bounded.

Now, we will show that $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$:

$$
U(t, r)U(r, s)z = \sum_{n=1}^{\infty} U_n(t, r)P_n \left( \sum_{i=1}^{\infty} U_i(r, s)P_iz \right) \\
= \sum_{n=1}^{\infty} U_n(t, r)U_n(r, s)P_nz \\
= \sum_{n=1}^{\infty} U_n(t, s)P_nz \\
= U(s, t)z
$$

Next, we show that $U(t, s)$ is strongly continuous. To show that $U(t, s)$ is strongly continuous, we must show that

$$
\lim_{t \downarrow 0} \|U(t, s)z - z\| = 0.
$$
Estimating $\|U(t, s)z - z\|^2$ yields:

\[
\begin{align*}
\|U(t, s)z - z\|^2 &= \left\| \sum_{n=1}^{\infty} U_n(t, s)P_nz - P_nz \right\|^2 \\
&\leq \sum_{n=1}^{\infty} \| (U_n(t, s) - I) P_nz \|^2 \\
&\leq \sum_{n=1}^{\infty} \| (U_n(t, s) - I) \|^2 \|P_nz\|^2
\end{align*}
\]

Let $\epsilon > 0$ be given. Then for $J$ large enough and $t$ small enough, we have,

\[
\begin{align*}
\sum_{n=J+1}^{\infty} \| (U_n(t, s) - I) \|^2 \|P_nz\|^2 &\leq \tilde{g}(t, s) \sum_{J+1}^{\infty} \|P_nz\|^2 \leq \epsilon \\
\end{align*}
\]

for a continuous nonnegative function $\tilde{g}(t, s)$. So,

\[
\begin{align*}
\sum_{n=1}^{\infty} \| (U_n(t, s) - I) \|^2 \|P_nz\|^2 &= \sum_{n=1}^{J} \| (U_n(t, s) - I) \|^2 \|P_nz\|^2 + \sum_{n=J+1}^{\infty} \| (U_n(t, s) - I) \|^2 \|P_nz\|^2 \\
&\leq \sup_{n \leq J} \| (U_n(t, s) - I) \|^2 \sum_{n=1}^{J} \|P_nz\|^2 + \epsilon.
\end{align*}
\]

As $t \downarrow 0$, $\sup_{n \leq J} \| (U_n(t, s) - I) \|^2 \to 0$. Thus, $\lim_{t \downarrow 0} \|U(t, s)z - z\| = 0$ and so $U(t, s)$ is strongly continuous.

Let $A(t)$ be the generator of this evolution operator. Then, from Definition 2.2, we have for all $z \in D$

\[
A(t)z = \lim_{h \to 0^+} \frac{U(t + h, t)z - z}{h} = \lim_{h \to 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t + h, t) - I)}{h} P_nz.
\]
Therefore,

\[ P_m A(t) z = P_m \left( \lim_{h \to 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t+h,t) - I)}{h} P_n z \right) \]

\[ = \lim_{h \to 0^+} \frac{(U_m(t+h,t) - I)}{h} P_m z = A_m(t) P_m z. \]

Hence,

\[ A(t) z = \sum_{n=1}^{\infty} P_n A(t) z = \sum_{n=1}^{\infty} A_n(t) P_n z, \]

and

\[ D \subset \{ z \in Z : \sum_{n=1}^{\infty} \| A_n(t) P_n z \|^2 < \infty, \ t \in [0,T] \}. \]

Now, suppose \( z \in \{ z \in Z : \sum_{k=1}^{\infty} \| A_k(t) P_k z \|^2 < \infty \ t \in [0,T] \} \). Then \( \sum_{k=1}^{\infty} \| A_k(t) P_k z \|^2 < \infty, t \in [0,T] \) and \( y = \sum_{k=1}^{\infty} A_k(t) P_k z \in Z. \)

Next, if we set \( z_n = \sum_{k=1}^{n} P_k z \), then \( z_n \in D \) and \( A(t) z_n = \sum_{k=1}^{n} A_k(t) P_k z. \)

Hence, \( \lim_{n \to \infty} z_n = z \) and \( \lim_{n \to \infty} A(t) z_n = y \), and since \( A(t) \) is a closed linear operator we get that \( z \in D \) and \( Az = y. \)

Proof of (c). It is equivalent to prove the following

\[ \rho(A(t)) = \bigcap_{n=1}^{\infty} \rho(\bar{A}_n(t)). \]

We shall prove that \( \rho(A(t)) \subset \bigcap_{n=1}^{\infty} \rho(\bar{A}_n(t)) \). In fact, let \( \lambda \) be in \( \rho(A(t)) \). Then \( (\lambda - A(t))^{-1} : Z \to D \) is a bounded linear operator. We need to prove that

\[ (\lambda - \bar{A}_m(t))^{-1} : \mathcal{R}(P_m) \to \mathcal{R}(P_m) \]
exists and is bounded for \( m \geq 1 \). Suppose that \((\lambda - \bar{A}_m(t))^{-1}P_m z = 0\). Then

\[
(\lambda - \mathcal{A}(t)) P_m z = \sum_{n=1}^{\infty} (\lambda - A_n) P_n P_m z = (\lambda - A_m) P_m z = (\lambda - \bar{A}_m) P_m z = 0.
\]

Which implies that, \( P_m z = 0 \). So, \((\lambda - \bar{A}_m(t))\) is one to one.

Now, given \( y \) in \( \mathcal{R}(P_m) \) we want to solve the equation \((\lambda - \bar{A}_m)w = y\). In fact, since \( \lambda \in \rho(A(t)) \) there exists \( z \in Z \) such that

\[
(\lambda - \mathcal{A}(t)) z = \sum_{n=1}^{\infty} (\lambda - A_n) P_n z = y.
\]

Then, applying \( P_m \) to the both side of this equation we obtain

\[
P_m(\lambda - \mathcal{A}(t)) z = (\lambda - A_m) P_m z = (\lambda - \bar{A}_m) P_m z = P_m y = y.
\]

Therefore, \((\lambda - \bar{A}_m(t)) : \mathcal{R}(P_m) \rightarrow \mathcal{R}(P_m)\) is a bijection, and since \( \mathcal{R}(P_m) \) is a closed, it is a Banach space. So, we can invoke the Open Mapping Theorem to conclude that \((\lambda - \bar{A}_m(t)) : \mathcal{R}(P_m) \rightarrow \mathcal{R}(P_m)\) exists and is a bounded linear operator. Hence, \( \lambda \in \rho(A_m(t)) \) for all \( m \geq 1 \). We have proved that

\[
\rho(A(t)) \subset \bigcap_{n=1}^{\infty} \rho(\bar{A}_n(t)) \iff \bigcup_{n=1}^{\infty} \sigma(\bar{A}_n) \subset \sigma(A(t)).
\]
Theorem 3.2. If $U(t,s), 0 \leq s \leq t \leq T$ is the evolution operator given by Lemma 3.1, $U(t,s)z \in D$ for all $z \in D$ and $0 \leq s \leq t \leq T$. Moreover, the Cauchy problem

$$
\begin{cases}
  z'(t) = A(t)z(t), & t > 0 \\
  z(s) = z_0, & z_0 \in D, \ 0 \leq s < t,
\end{cases} \quad (12)
$$

has the unique solution

$$
z(t) = U(t,s)z_0, \ t \geq s. \quad (13)
$$

Proof. Due to Lemma 2.1 and Theorem 2.3, we only need to verify the first statement of this theorem. Let us consider $z \in D$. Then $\sum_{n=1}^{\infty} \|A_n(t)P_nz\|^2 < \infty$, for all $0 \leq t \leq T$ and so

$$
\sum_{n=1}^{\infty} \|A_n(t)U_n(t,s)P_nz\|^2 \leq g(t,s)^2 \sum_{n=1}^{\infty} \|A_n(t)P_nz\|^2 < \infty,
$$

for all $0 \leq s \leq t \leq T$. Hence, $U(t,s)z \in D$ for all $z \in D$ and $0 \leq s \leq t \leq T$. \[ \square \]

4. Applications

In this section we shall present some applications of Lemma 3.1 to prove the existence and the uniqueness of the some time-dependent systems of partial differential equations. Namely, we will discuss the wave equation with time dependent damping and a thermoelastic plate equation with time dependent coefficients.
4.1. Time-Dependent Generalized Damped Wave Equation.

\[ \ddot{w} + \eta(t)\dot{w} + \gamma A^\beta w = 0 \quad t \geq 0, \quad (14) \]

where \( \gamma > 0, \eta(t) > 0 \) is a continuous function, \( \beta \geq 0 \) and \( A : D(A) \subset X \to X \) is a positive definite self-adjoint unbounded linear operator in \( X \) with compact resolvent. We consider the operator

\[ A(t) = \begin{bmatrix} 0 & I_X \\ -\gamma A^\beta & -\eta(t) \end{bmatrix}, \quad (15) \]

which corresponds to this equation written as a first order system in the space \( D(A^{\beta/2}) \times X \). Applying Lemma 3.1, one can easily verify that \( A(t) \) generates an evolution operator \( \{ U(t,s) : 0 \leq s \leq t \} \) on \( D(A^{\beta/2}) \times X \).

4.2. Time-Dependent Thermoelastic Equation. Using the Lemma 3.1 one can study the existence and the uniqueness of the following time-dependent thermoelastic plate equation with Dirichlet boundary condition

\[
\begin{cases}
    w_{tt} + \Delta^2 w + \alpha(t)\Delta \theta = 0, & t \geq 0, \quad x \in \Omega, \\
    \theta_t - \beta(t)\Delta \theta - \alpha(t)\Delta w_t = 0, & t \geq 0, \quad x \in \Omega, \\
    \theta = w = \Delta w = 0, & t \geq 0, \quad x \in \partial \Omega,
\end{cases}
\quad (16)
\]

where \( \alpha(t) > 0, \beta(t) > 0 \) is a continuous function, \( \Omega \) is a sufficiently regular bounded domain in \( \mathbb{R}^N \), and \( w, \theta \) denote the vertical deflection and the temperature of the plate respectively.
The derivation of the time independent \((\alpha(t) = \alpha, \beta(t) = \beta = \text{constant})\) thermoelastic plate equation can be found in J. Lagnese [3], where the author discussed stability of various plate models. J.U. Kim [1](1992) studied the system (??) with the following homogeneous Dirichlet boundary condition

\[
\theta = \frac{\partial w}{\partial \eta} = w = 0, \quad \text{on} \quad \partial \Omega.
\]

Also, one can see the work done in [6]. As example 1.1 we consider the Hilbert space \(X = L_2(\Omega)\) and define \(A : D \to X\) to be the linear unbounded operator:

\[
A\phi = -\triangle \phi,
\]

where the domain of \(A\) is \(D = H^1_0(\Omega) \cap H^2(\Omega)\). The spectral properties of \(A\) are well known and \(A\) can be decomposed as:

\[
Ay = \sum_{j=1}^{\infty} \lambda_j E_j y,
\]

where \(\{\lambda_j\}_{j=1}^{\infty}\) are the eigenvalues of the Laplacian and \(\{E_j\}\) is the complete orthogonal sequence of projections,

\[
E_j x = \sum_{k=1}^{\gamma_j} <\phi_{j,k}, x> \phi_{j,k},
\]

defined through the eigenvectors \(\{\phi_{j,k}\}_{k=1}^{\gamma_j}\) corresponding to eigenvalue \(\lambda_j\) for each \(j\).

The fractional powered spaces \(X^r\) are given by:

\[
X^r = D(A^r) = \{ x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \}, \quad r \geq 0,
\]
with the norm
\[
\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,
\]
and
\[
A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \quad (17)
\]

Also, for \( r \geq 0 \) we define \( Z_r = X^r \times X \times X \), which is a Hilbert Space with the norm given by:
\[
\left\| \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2 + \|\theta\|^2,
\]

Hence, the equation (16) can be rewritten as an abstract system of ordinary differential equation in \( Z_1 = X^1 \times X \times X \) as follows
\[
\begin{cases}
w' = v \\
v' = -A^2 w + \alpha(t) A \theta \\
\theta' = -\beta(t) A \theta - \alpha(t) A v.
\end{cases} \quad (18)
\]

Finally, system (16) can be rewritten as first order system of ordinary differential equations in the Hilbert space \( Z_1 = X^1 \times X \times X \) as follows
\[
z' = A(t)z \quad z \in Z_1, \quad t \geq 0, \quad (19)
\]
where

\[ z = \begin{bmatrix} w \\ v \\ \theta \end{bmatrix}, \]

and

\[ A(t) = \begin{bmatrix} 0 & I_x & 0 \\ -A^2 & 0 & \alpha(t)A \\ 0 & -\alpha(t)A & -\beta(t)A \end{bmatrix}, \]

(20)
is an unbounded linear operator with common domain

\[ D(A(t)) = \{ w \in H^4(\Omega) : w = \Delta w = 0 \} \times D(A) \times D(A). \]

Under some additional condition on the coefficients \( \alpha(t) \) and \( \beta(t) \) which were provided in [5] one can use Lemma 3.1 to show that the operator \( A(t) \) given by (20), is the infinitesimal generator of an evolution operator \( \{ U(t, s) : 0 \leq s \leq t \leq T \} \) on the space \( Z_1 \).

**References**


