Analysis of a nonautonomous Nicholson Blowfly model

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Abstract

Most dynamic models describing population evolution contain one or more parameters. The parameters are treated as fixed constants and qualitative results, such as stability of equilibria, are calculated using this assumption. In reality, however, the parameters are mathematically evaluated by statistical methods in which the error is decreased over a number of calculations. Therefore, the parameter is a sequence converging to the actual parameter value as time goes to infinity. In this article we consider the $k$th-order discrete Nicholson Blowfly model, $N_{n+1} = F(P, \delta, N_n, \ldots, N_{n-k})$ where $\delta$ and $P$ are parameters. For a particular range of parameter values, global stability results are well known. The general form of the discrete dynamical system is now rewritten as $N_{n+1} = F(P_n, \delta_n, N_n, \ldots, N_{n-k})$ where $P_n$ and $\delta_n$ converge to the parametric values $P$ and $\delta$. We show that when the parameters are replaced by sequences, the stability results of the original system still hold. This technique may be of general interest to those studying evolutionary systems in which the parameters are not fundamental constants but sequences. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

Many dynamical systems that model biological phenomena contain several parameters. Biologists are tasked to determine the exact parameter values in order to use the model for prediction purposes. Unfortunately, in the real world, parameters are not fixed constants. Typically, the parameters are estimated using statistical methods and...
at each stage in time the estimate will be improved. Therefore, the parameters in a model are actually sequences that converge to a constant parameter value as time goes to infinity.

This article considers the discrete Nicholson Blowfly model,

\[ N_{n+1} - N_n = -\delta N_n + P N_{n-k} e^{-aN_{n-1}} , \tag{1} \]

which has been studied in many recent papers [2,3]. We extend results on global asymptotic behavior to the nonautonomous Nicholson model,

\[ N_{n+1} - N_n = -\delta N_n + P N_{n-k} e^{-aN_{n-1}} , \tag{2} \]

where \( \lim_{n \to \infty} P_n = P \) and \( \lim_{n \to \infty} \delta_n = \delta \). The implications of our results guarantee the robustness of solutions to Eq. (1).

It may seem at first glance that any discrete dynamical system where parameters are replaced with convergent sequences should have the same asymptotic behavior. To show that the dynamics can change, we present two counterexamples in the following subsection.

1.1. Two counterexamples

Example 1.1. For \( n \geq 1 \) consider the discrete system,

\[ x_{n+1} = x_n + \frac{1}{n} . \tag{3} \]

As \( n \to \infty \), we obtain the autonomous limiting equation,

\[ y_{n+1} = y_n . \tag{4} \]

The general solution for Eq. (4) is \( y_n = y_1 \). That is, every initial condition in \( \mathbb{R} \) is a fixed point. However, the nonautonomous equation (3), has the general solution \( x_n = x_1 + \frac{1}{2} + \cdots + 1/n \), which is the initial condition plus the partial sum of the harmonic series. Therefore, every solution of Eq. (3) goes to infinity.

The second example, which was obtained by discretizing a differential equation in [4], shows that a semi-stable fixed point can become a global repeller. This example shows that dynamics can change even if the convergence is exponential.

Example 1.2. Let \( \alpha > 0 \) and for \( n \geq 0 \) consider the nonautonomous discrete system,

\[ x_{n+1} = x_n + |x_n| + e^{-\alpha n} . \tag{5} \]

The limiting system for Eq. (5) is,

\[ y_{n+1} = y_n + |y_n| . \tag{6} \]

The general solution for Eq. (6) if \( y_0 > 0 \) is given by \( y_n = 2^n y_0 \). If \( y_0 \leq 0 \), then \( y_n = 0 \) for \( n \geq 1 \). Therefore, the solution to (6) converges to infinity if the initial condition is positive and 0 if the initial condition is non-positive and so, the zero equilibrium is semi-stable.
Table 1
Parameter and sequence values for numerical results

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>$P$</th>
<th>$\delta$</th>
<th>$P_n$</th>
<th>$\delta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.2</td>
<td>$P(1 + 3.6(-1)^n)/n$</td>
<td>$\delta(1 + 9</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.4</td>
<td>$P(1 + 3.6(-1)^n)/n$</td>
<td>$\delta(1 +</td>
</tr>
</tbody>
</table>

Now for Eq. (5), if $x_0 > 0$, the solution, $x_n \geq c2^n$ where $c > 0$, and so $x_n$ converges to infinity. But if $x_0 \leq 0$, $x_1 = 1 > 0$. So in one iteration, $x_n$ becomes positive and hence also converges to infinity. Thus, the zero equilibrium value is now a global repeller.

1.2. The model and stability results

In this section we will develop the intuition behind (1) that was introduced in [1]. In [1], Eq. (1) was presented as a differential delay equation which has been studied extensively in the literature [2]. Analytically, differential delay equations are difficult to manage and therefore recent articles have examined the model as the difference equation (1) [2,3].

The state variable, $N_n$, in Model (1), represents the number of sexually mature blowflies in cycle $n$ as a closed system of the mature flies surviving from the previous cycle plus the flies that have survived from the previous $k$th cycle. Specifically,

$$PN_{n-k}e^{-\delta N_{n-k}}$$

represents the number of mature flies that were born in the $(n-k)$th cycle and survived to maturity in the $n$th cycle. The parameter, $P$, is positive and represents the maximum possible per capita egg production and $1/a > 0$ is the population size at which there is maximum reproductive success. The parameter, $\delta \in [0,1]$, is the death rate per fly per cycle and so the term $(1 - \delta)N_n$ models the number of older flies that survive the $n$th cycle. Notice that for the population to grow we must have $P > \delta$ (Table 1).

There are two equilibria for the autonomous Nicholson model, $N_0 = 0$, which represents extinction, and $\bar{N} = 1/a \ln(P/\delta)$. It was shown in [3] that when $P < \delta$, $N_0 = 0$ is a global attractor. Biologically, when $P < \delta$, the reproductive rate is less than the death rate and so the population will go to extinction. Also in [3], it was proved when $P > \delta$ and

$$((1 - \delta)^{-(k+1)} - 1) \ln(P/\delta) \leq 1,$$  \hspace{1cm} (7)

then $\bar{N}$ will be a global attractor. Kocic and Ladas obtained results showing if $P$ became too large, the solution would oscillate about $\bar{N}$ [2].

The question this article addresses is whether $N_0$ and $\bar{N}$ remain global attractors if the death rate $\delta$ and the survival rate $P$ are replaced with convergent sequences. Our work will allow for certain cycles to have a zero survival rate and or a zero reproductive
rate yet in the long-term converge to positive parameters. In Section 2 we will provide numerical results showing that even when \( P_n \) and \( \delta_n \) oscillate, the global asymptotic results still hold. Section 3 will provide analytical proofs to the conjectures developed in Section 2.

2. Numerical results

We ran many numerical experiments for Eq. (2) where \( \{P_n\} \) and \( \{\delta_n\} \) were chosen to be monotone, or oscillating sequences. Our numerical data visually represent the global asymptotic results in the next section. The only conditions we applied to the sequential parameters are that \( \lim_{n \to \infty} \delta_n = \delta > 0 \) and \( \lim_{n \to \infty} P_n = P > 0 \). The sequence terms were restricted to \( \delta_n, P_n \in [0, \infty) \). Fig. 1 shows that if the limiting values of the sequential parameters satisfy, \( \delta < P \) and the inequality \( ((1 - \delta)^{(k+1)} - 1) \ln \left( \frac{\delta}{\delta} \right) \leq 1 \) then the autonomous and nonautonomous solutions both converge to the same value, \( \bar{N} \). It is surprising that the condition, \( P_n > \delta_n \), does not hold term by term. Fig. 2 shows that \( \delta_n \) is larger than \( P_n \) in the initial terms of the sequence. This means even if initially the blowfly environment is harsh, they can still recover and head to a stable population value.

Fig. 3 shows that both the autonomous solution and the nonautonomous solution converge to extinction if \( P < \delta \). The sequences chosen for this particular example show that even if there are particular cycles where \( P_n > \delta_n \), it is not enough to hold off the extinction of the population (see Fig. 4).
Fig. 2. The sequences, \( \{ \delta_n \} \) and \( \{ P_n \} \), in Fig. 1 do not satisfy \( \delta_n < P_n \) term by term.

Fig. 3. Nonautonomous and autonomous solutions for oscillating \( \delta_n \) and \( P_n \) and \( P < \delta \).

3. Analytical results

We will now prove several properties of the nonautonomous Nicholson Blowfly model where the parameters satisfy the conditions,

\[ \delta > 0, P > 0 \quad \text{and} \quad a > 0, \]

and the varying constants satisfy \( \delta_n \geq 0 \) and \( P_n \geq 0 \) for all \( n \geq 0 \). These conditions imply that although \( \delta \) and \( P \) must be strictly positive, a finite number of zero terms in
the sequences are allowed. Under these parameter conditions, Eq. (2) yields bounded and positive solutions as stated in the following proposition.

**Proposition 1.** Consider system (2) with \( \lim_{n \to \infty} p_n = P \) and \( \lim_{n \to \infty} \delta_n = \delta \) where \( \delta \in (0,1) \), \( P \in (0,\infty) \) and \( \delta_n \in [0,1] \), \( P_n \in [0,\infty) \) for each \( n \geq 0 \). Assume further that the initial conditions \( N_{-k}, \ldots, N_0 \) are nonnegative. Then the solution, \( \{N_n\}_{n=1}^\infty \) is bounded and nonnegative.

**Proof.** First, assume that \( N_{n-k}, \ldots, N_n \) are nonnegative. Then

\[
N_{n+1} = (1 - \delta_n)N_n + P_nN_{n-k}e^{-aN_{n-k}} \geq 0,
\]

which proves the nonnegativity result by induction.

Now, assume that \( \{N_n\} \) is not bounded. Then there is a subsequence, \( \{N_{n_i}\} \), such that \( \lim_{i \to \infty} N_{n_i} = \infty \) and \( N_{n_{i+1}} - N_{n_i} \geq 0 \). Using Eq. (2), we have that

\[
0 \leq -\delta_{n_i}N_{n_i} + P_{n_i}N_{n_i-k}e^{-aN_{n_i-k}}.
\]

For \( i \) large enough, \( P_{n_i} > 0 \) and so,

\[
\frac{\delta_{n_i}}{P_{n_i}}N_{n_i} < N_{n_i-k}e^{-aN_{n_i-k}} \leq N_{n_i-k}.
\]

Taking the limit of both sides of the inequality indicates that \( N_{n_i-k} \to \infty \). However, this implies that

\[
0 \leq \delta_{n_i}N_{n_i} \leq P_{n_i}N_{n_i-k}e^{-aN_{n_i-k}} \to 0
\]

as \( i \to \infty \) and hence \( \lim_{i \to \infty} N_{n_i} = 0 \) which creates a contradiction. Therefore, the assumption is false and \( \{N_n\} \) is bounded. \( \square \)
As stated in Section 1.2, zero is a globally asymptotically stable solution for Eq. (1) when $P \leq \delta$ [3]. We first prove a short proposition stating that for large $n$ and $P < \delta$, $\{N_n\}$ cannot be a strictly increasing sequence. Then the following theorem guarantees zero is globally asymptotically stable for Eq. (2) whenever $P < \delta$.

**Proposition 2.** Suppose that $\delta_n \in [0, 1]$ and $P_n \in [0, \infty)$ for all $n \geq 1$. Assume that $\lim_{n \to \infty} \delta_n = \delta \in (0, 1)$ and $\lim_{n \to \infty} P_n = P > 0$ and that $P < \delta$. Let $\epsilon$ be so small and $N$ be so large such that $P_n < \delta - \epsilon < \delta_n$ for $n \geq N$. Then the solution to Eq. (2) cannot be a strictly increasing sequence for $n \geq N$.

**Proof.** To see this first assume on the contrary that $\{N_n\}$ is strictly increasing. Then there is an $M > 0$ such that $N_{n+1} > M$ and $N_{n-k}, \ldots, N_n < M$ (see Fig. 5).

So

$$M < (1 - \delta_n)N_n + P_n N_{n-k} e^{-aN_{n-1}} \leq (1 - \delta_n)M + \delta_n M = M,$$

which is a contradiction. So $\{N_n\}$ cannot be increasing for $n \geq N$. □

**Theorem 3.** Suppose that $\delta_n \in [0, 1]$ and $P_n \in [0, \infty)$ for all $n \geq 1$. Assume that $\lim_{n \to \infty} \delta_n = \delta$ and $\lim_{n \to \infty} P_n = P$ and that $P \leq \delta$. Then all forward orbits of $\{N_n\}$ with $N_{-k}, \ldots, N_0 \in [0, \infty)$ converge to zero.

**Proof.** First, assume that $N_n$ is either oscillating or increasing (recall that we proved $N_n$ cannot be strictly increasing for $P < \delta$). Assume on the contrary that $0 < \mu = \limsup N_n$. Then we can find a subsequence $\{N_{n_i}\}$ such that $N_{n_i+1} - N_{n_i} > 0$ and $\lim_{n \to \infty} N_{n_i} = \mu$. Let $\epsilon > 0$ be given so that for all $n \geq N$, $P_n \leq \delta_n + \epsilon$. Then for $n_i \geq N$,

$$0 \leq N_{n_i+1} - N_{n_i} = -\delta_n N_{n_i} + P_{n_i} N_{n_i-k} e^{-aN_{n_i-k}}$$

$$\Rightarrow \delta_n N_{n_i} \leq P_{n_i} N_{n_i-k} e^{-aN_{n_i-k}} \leq (\delta_n + \epsilon) N_{n_i-k}.$$  

(8)

So $\mu \leq \limsup N_{n_i-k} \leq \limsup N_n = \mu$. This implies that $\limsup N_{n_i-k} = \mu$.

Taking the lim sup of both sides of inequality (8) yields

$$\delta \mu \leq P \mu e^{-a\mu} \leq \delta \mu e^{-a\mu},$$

where $\delta$ and $P$ are constants. Thus $\mu = 0$, and the theorem is proved.

Fig. 5. An increasing sequence has to eventually get above some $M > 0$. 

$\text{n}$
which implies that $\mu < \mu e^{-\alpha_1}$. This creates a contradiction. Therefore, our assumption was false and $\mu = 0$.

Finally, assume that $\{N_n\}$ is a non-increasing sequence for $n \geq N$. Then the $\lim_{n \to \infty} N_n$ exists. Let $N^0 = \lim_{n \to \infty} N_n$. Taking the limit of both sides of Eq. (2) produces the algebraic equation,

$$N^0 = (1 - \delta)N^0 + N^0 P e^{-a N^0}.$$  

The algebraic equation has two solutions, $N^0 = 0$ and $N^0 = 1/a \ln (P/\delta)$. Since $P < \delta$, the second value is negative and hence, $N^0 = 0$. This completes the proof.

The next theorem provides parameter regions for the stability of the non-zero value, $\hat{N}$.

**Theorem 4.** Assume that $P > \delta$ and $\lim_{n \to \infty} \delta_n = \delta$ and $\lim_{n \to \infty} P_n = P$. Assume further that

$$((1 - \delta)^{-k-1} - 1) \ln \left( \frac{P}{\delta} \right) \leq 1.$$  

Then every forward orbit of Eq. (2) converges to $\hat{N} = 1/a \ln (P/\delta)$.

**Proof.** To begin the argument, we first make the change of variables:

$$N_n = \hat{N} + \frac{1}{a} x_n$$  

to get,

$$x_{n+1} - x_n = -\delta_n x_n - a\delta \hat{N} (d_n - \rho_n e^{-x_n}) + \rho_n \delta x_{n-k} e^{-x_{n-k}},$$  

where $d_n = \delta_n / \delta$ and $\rho_n = P_n / P$.

**Case 1:** Assume that there is an $N > 0$ such that $\{x_n\} > 0$ for all $n \geq N$. This case represents the situation where $N_n$ is eventually higher than $\hat{N}$ and stays above $\hat{N}$. We claim that in this case $\lim_{n \to \infty} x_n = 0$. Suppose first that $x_n$ is either increasing or oscillating for $n \geq N$ and assume on the contrary that $0 < \mu = \limsup_{n \to \infty} x_n$. Then we can find a subsequence $\{x_{n_i}\}$ such that $x_{n_{i+1}} - x_{n_i} \geq 0$ for all $n_i \geq N$, and $\lim_{n \to \infty} x_{n_i} = \mu$. Therefore,

$$0 \leq -\delta_n x_n - a\delta \hat{N} (d_n - \rho_n e^{-x_n}) + \rho_n \delta x_{n-k} e^{-x_{n-k}}.$$  

So,

$$\delta_n x_n + a\delta_n \hat{N} \leq a\delta \hat{N} \rho_n e^{-x_n} + \rho_n \delta x_{n-k} e^{-x_{n-k}} \leq a\delta \hat{N} \rho_n + \rho_n \delta x_{n-k}.$$  

(10)

Taking the $\limsup$ of both sides yields

$$\delta \mu + a\delta \hat{N} \leq a\delta \hat{N} + \delta \limsup_{n_i} x_{n_i-k}.$$
So, \( \mu \leq \limsup_n x_{n-k} \leq \limsup_n x_n = \mu \), which implies that
\[
\limsup_{n} x_{n-k} = \mu.
\]

But then we have
\[
\delta \mu + a \delta \tilde{N} \leq a \delta \tilde{N} e^{-\mu} + \delta \mu \leq a \delta \tilde{N} + \delta \mu
\]
and hence, \( \mu \leq \mu e^{-\mu} \). Since \( \mu > 0 \), this creates a contradiction and hence our assumption was false and \( \mu = 0 \). Therefore, in this case, \( \lim_{n \to \infty} x_n = 0 \).

Now since \( \{x_n\} \) is bounded below by zero in this case, if it is decreasing, it has a limit. Let \( \tilde{x} \) be the limit. Taking the limit of both sides of Eq. (9) yields the algebraic equation,
\[
0 = -\delta \tilde{x} - a \delta \tilde{N} (1 - e^{-\tilde{x}}) + \delta \tilde{N} e^{-\tilde{x}}.
\]

The algebraic equation has two solutions, \( \tilde{x} = 0 \) and \( -a \tilde{N} \). The second solution is negative since \( \delta < P \) so \( \tilde{x} = 0 \) and so \( \lim_{n \to \infty} x_n = 0 \).

The proof of the case where there is an \( N > 0 \) such that \( x_n < 0 \) for all \( n \geq N \) is similar to Case 1.

Case 2: Assume that \( \{x_n\} \) oscillates about zero.

Set \( \mu = \limsup_n x_n \) and \( \lambda = \liminf_n x_n \). We will show that \( \lambda = \mu = 0 \).

Since \( \{x_n\} \) oscillates, there is a subsequence \( \{x_{n_i}\} \) such that \( x_{n_i} < 0 \) and \( x_{n_i+1} \geq 0 \).

Let \( \{M_i\} \) and \( \{m_i\} \) be two sequences of positive integers such that
\[
x_{M_i} = \max\{x_j : n_i < j < n_{i+1}\},
\]
and
\[
x_{m_i} = \min\{x_j : n_i < j < n_{i+1}\}.
\]
Clearly, \( x_{M_i} \geq 0 \) and \( x_{m_i} \leq 0 \). Choose the interval \((n_i, n_{i+1})\) spread out large enough so that
\[
x_{M_i} - x_{M_i-1} \geq 0 \quad \text{and} \quad \limsup_i x_{M_i} = \mu,
\]
\[
x_{m_i} - x_{m_i-1} \leq 0 \quad \text{and} \quad \liminf_i x_{m_i} = \lambda.
\]

Finally, let \( n^* \) be so large and \( \varepsilon > 0 \) be so small so that \( \delta - \varepsilon < \delta_n < \delta + \varepsilon \) and \( 1 - (\delta + \varepsilon) > 0 \) for all \( n \geq n^* \). In addition, we want \( \lambda - \varepsilon \leq x_n \leq \mu + \varepsilon \) for all \( n \geq n^* \).

From here on, our arguments will assume that \( n \geq n^* \). The proof will be separated into three steps to simplify the reading.

Step 1: We will show there exists a \( p_i \) with \( \max\{n_i, M_i - 1 - k\} \leq p_i \leq M_i \) such that \( x_{p_i} \leq 0 \) and \( x_{p_i+1} \geq 0 \).

Likewise, a similar argument will show that there is a \( q_i \) such that
\[
\max\{n_i, m_i - 1 - k\} \leq q_i \leq m_i,
\]
with \( x_{q_i} \geq 0 \) and \( x_{q_i+1} \leq 0 \).
If \( n_i \geq M_i - k - 1 \) then we are done because \( p_i = n_i \). Assume that \( n_i < M_i - 1 - k \) and assume on the contrary that \( p_i \) does not exist. Then \( x_j > 0 \) for \( M_i - 1 - k \leq j \leq M_i \). Since \( x_{M_i} - x_{M_i-1} \geq 0 \) we have,

\[
\delta_{M_i-1} x_{M_i-1} \leq -a \delta N (d_{M_i-1} - \rho_{M_i-1} e^{-x_{M_i-1}}) + \rho_{M_i-1} \delta x_{M_i-1} e^{-x_{M_i-1}}
\]

so,

\[
x_{M_i-1} + a \tilde{N} \leq a \delta_{M_i-1} (a \tilde{N} + x_{M_i-1}) e^{-x_{M_i-1}}.
\]

Now,

\[
x_{M_i} + a \tilde{N} = (1 - \delta_{M_i-1}) x_{M_i-1} - a \delta N (d_{M_i-1} - \rho_{M_i-1} e^{-x_{M_i-1}}) + \rho_{M_i-1} \delta x_{M_i-1} e^{-x_{M_i-1}} + a \tilde{N}
= (1 - \delta_{M_i-1}) \frac{\delta}{\delta_{M_i-1}} \rho_{M_i-1} (a \tilde{N} + x_{M_i-1}) e^{-x_{M_i-1}} + a \tilde{N}
\leq \frac{\delta}{\delta_{M_i-1}} \rho_{M_i-1} (a \tilde{N} + x_{M_i-1}) e^{-x_{M_i-1}}.
\]

At this time, assume that \( x_{M_i-1-k} \) converges to zero. By taking the limit of both sides of Inequality (11), we get,

\[
\mu + a \tilde{N} \leq a \tilde{N},
\]

which reduces to \( \mu \leq 0 \). But \( \mu \geq 0 \), so we conclude that \( \mu = 0 \). We will return to this particular conclusion towards the end of Step 2.

Now consider on the other hand that \( x_{M_i-1-k} \) does not converge to zero. Then there is a subsequence of \( x_{M_i-1-k} \) that is bounded uniformly away from zero. From here on, our discussion will be restricted to this further subsequence. Since \( x_{M_i-1-k} \) is bounded uniformly away from zero, we can find an \( \tilde{\varepsilon} > 0 \) such that

\[
e^{-x_{M_i-1-k}} < 1 - \tilde{\varepsilon}
\]

and \( \tilde{\varepsilon} \) does not depend on \( M_i \). Since \( \frac{\delta}{\delta_{M_i-1}} \rho_{M_i-1} \) converges to one, we have for \( M_i \) large enough, \( \frac{\delta}{\delta_{M_i-1}} \rho_{M_i-1} < 1 + \tilde{\varepsilon} \). So for \( M_i \) large enough,

\[
x_{M_i} + a \tilde{N} < (a \tilde{N} + x_{M_i-1-k}) (1 + \tilde{\varepsilon})(1 - \tilde{\varepsilon})
= (a \tilde{N} + x_{M_i-1-k}) (1 - \tilde{\varepsilon}^2)
< a \tilde{N} + x_{M_i-1-k}.
\]

This creates a contradiction since \( x_{M_i} = \max \{ x_j : n_i \leq j \leq n_i+1 \} \) and so our assumption was false and \( p_i \) exists.
Step 2: We will show that $\lambda$ and $\mu$ satisfy the inequalities

$$\mu \leq e^{-\lambda} - 1,$$

$$\lambda \geq e^{-\mu} - 1.$$

First, observe that $x_{n-k}e^{x_{n-k}} < \mu + \epsilon$ (consider $x_{n-k} \geq 0$ and then $x_{n-k} < 0$). Multiply Eq. (9) through by $(1 - (\delta - \epsilon))^{-(n+1)}$ to obtain:

$$(1 - (\delta - \epsilon))^{-(n+1)}x_{n+1} - (1 - \delta_n)(1 - (\delta - \epsilon))^{-(n+1)}x_n = -a\delta N(d_n - \rho_n e^{-x_n-\epsilon})(1 - (\delta - \epsilon))^{-(n+1)}$$

$$+ \rho_n\delta(1 - (\delta - \epsilon))^{-(n+1)}x_{n-k}e^{-x_{n-k}}. \quad (12)$$

Summing Equation (12) from $n = p_1$ to $n = M_{i-1}$ yields:

$$\sum_{p_1}^{M_{i-1}} (1 - (\delta - \epsilon))^{-(n+1)}x_{n+1} - \sum_{p_1}^{M_{i-1}} (1 - \delta_n)(1 - (\delta - \epsilon))^{-(n+1)}x_n$$

$$= \sum_{p_1}^{M_{i-1}} [a\delta N(d_n - \rho_n e^{-x_n-\epsilon})(1 - (\delta - \epsilon))^{-(n+1)}$$

$$+ \rho_n\delta(1 - (\delta - \epsilon))^{-(n+1)}x_{n-k}e^{-x_{n-k}}]. \quad (13)$$

Observe that the left hand side of Equation (13) satisfies the inequality,

$$\sum_{p_1}^{M_{i-1}} (1 - (\delta - \epsilon))^{-(n+1)}x_{n+1} - \sum_{p_1+1}^{M_{i-1}} (1 - \delta_n)(1 - (\delta - \epsilon))^{-(n+1)}x_n$$

$$-(1 - \delta_{p_1})(1 - (\delta - \epsilon))^{-p_1}x_{p_1}$$

$$\geq \sum_{p_1}^{M_{i-1}} (1 - (\delta - \epsilon))^{-(n+1)}x_{n+1} - \sum_{p_1}^{M_{i-1}} (1 - \delta_n)(1 - (\delta - \epsilon))^{-(n+1)}x_n$$

$$-(1 - \delta_{p_1})(1 - (\delta - \epsilon))^{-p_1}x_{p_1}$$

$$= (1 - (\delta - \epsilon))^{-M_{i-1}} - (1 - \delta_{p_1})(1 - (\delta - \epsilon))^{-p_1}x_{p_1}.$$ Therefore,

$$\sum_{n=p_1}^{M_{i-1}} -a\delta N(d_n - \rho_n e^{-x_n-\epsilon})(1 - (\delta - \epsilon))^{-(n+1)}$$

$$+ \sum_{n=p_1}^{M_{i-1}} -a\delta N(d_n - \rho_n e^{-x_n-\epsilon})(1 - (\delta - \epsilon))^{-(n+1)}$$

$$+ \rho_n\delta(1 - (\delta - \epsilon))^{-(n+1)}x_{n-k}e^{-x_{n-k}}.$$
\[
\sum_{n=0}^{M-1} a_n \delta(1-(\delta-\varepsilon))^{-(n+1)} x_{n-k} e^{-x_{n-k}} \\
\leq \sum_{n=0}^{M-1} -a \delta N (d_n - \rho_n e^{-x_{n-k}}) (1-(\delta-\varepsilon))^{-(n+1)} \\
+ \sum_{n=0}^{M-1} a_n \delta(1-(\delta-\varepsilon))^{-(n+1)} x_{n-k} e^{-x_{n-k}}. \tag{14}
\]

Since \(\rho_n\) and \(\delta_n\) converge to one, we have that 
\[1 - \varepsilon < \rho_n, \delta_n < 1 + \varepsilon\]
for \(n\) large enough. Consider the first term on the right hand side of Inequality (14). For each 
\[p_i \leq n \leq M-1\]
we have,
\[-a \delta N d_n (1-(\delta-\varepsilon))^{-(n+1)} + a \delta N \rho_n e^{-x_{n-k}} (1-(\delta-\varepsilon))^{-(n+1)}
\leq -a \delta N (1-\varepsilon) (1-(\delta-\varepsilon))^{-(n+1)} + a \delta N (1+\varepsilon) e^{-x_{n-k}} (1-(\delta-\varepsilon))^{-(n+1)}
= -a \delta N (1-(\delta-\varepsilon))^{-(n+1)} (1-\varepsilon - (1+\varepsilon) e^{-x_{n-k}})
\leq -a \delta N (1-(\delta-\varepsilon))^{-(n+1)} (1-\varepsilon - (1+\varepsilon) e^{-\lambda+\varepsilon}).\]

Taking into account that \(x_{n-k} e^{-x_{n-k}} < \mu + \varepsilon\), the second term on the right hand side of Inequality (14) is
\[
\sum_{n=p_i}^{M-1} a_n \delta(1-(\delta-\varepsilon))^{-(n+1)} x_{n-k} e^{-x_{n-k}}
\leq \sum_{n=p_i}^{M-1} a_n \delta(1-(\delta-\varepsilon))^{-(n+1)} (\mu + \varepsilon).
\]

Therefore, Inequality (14) can be reduced to,
\[
(1-(\delta-\varepsilon))^{-M} x_M
\leq \left[-a \delta N (1-\varepsilon - (1+\varepsilon) e^{-\lambda+\varepsilon}) + (1+\varepsilon) \delta (\mu + \varepsilon)\right] \\
\times \sum_{n=p_i}^{M-1} (1-(\delta-\varepsilon))^{-(n+1)}.
\]

The sum, \(\sum_{n=p_i}^{M-1} (1-(\delta-\varepsilon))^{-(n+1)}\), is a finite geometric series and can be calculated,
\[
\sum_{n=p_i}^{M-1} (1-(\delta-\varepsilon))^{-(n+1)} = \frac{1}{\delta - \varepsilon} ((1-(\delta-\varepsilon))^{-M} - (1-(\delta-\varepsilon))^{-p_i}).
\]
Substituting into Inequality (14) yields,

\[
(1 - (\delta - \varepsilon))^{-M_i} x_{M_i} \leq \frac{1}{\delta - \varepsilon} ((1 - (\delta - \varepsilon))^{-M_i} - (1 - (\delta - \varepsilon))^{-p_i} [(-a\delta \tilde{N}(1 - \varepsilon - (1 + \varepsilon)e^{-\lambda+i} + (1 + \varepsilon)\delta(\mu + \varepsilon)]] .
\]  
(15)

We divide Inequality (15) through by \((1 - (\delta - \varepsilon))^{-M_i}\) to obtain

\[
x_{M_i} \leq \frac{1}{\delta - \varepsilon} (1 - (1 - (\delta - \varepsilon))^{M_i-p_i})
\times [(-a\delta \tilde{N}(1 - \varepsilon - (1 + \varepsilon)e^{-\lambda+i} + (1 + \varepsilon)\delta(\mu + \varepsilon)] .
\]

Recall that \(M_i - p_i \leq k + 1\). Thus,

\[
x_{M_i} \leq \frac{1}{\delta - \varepsilon} (1 - (1 - (\delta - \varepsilon))^{k+1})
\times [(-a\delta \tilde{N}(1 - \varepsilon - (1 + \varepsilon)e^{-\lambda+i} + (1 + \varepsilon)\delta(\mu + \varepsilon)] .
\]  
(16)

Since \(\limsup \ x_{M_i} = \mu\), \(x_{M_i}\) has a subsequence converging to \(\mu\). Inequality (16) holds for the subsequence and so letting \(\varepsilon \to 0\) produces,

\[
\mu \leq \frac{1}{\delta} (-a\delta \tilde{N}(1 - e^{-\lambda}) + \delta\mu)(1 - (1 - \delta)^{k+1})
= (-a\tilde{N}(1 - e^{-\lambda}) + \mu)(1 - (1 - \delta)^{k+1})
= -a\tilde{N}(1 - e^{-\lambda})(1 - (1 - \delta)^{k+1}) + \mu(1 - (1 - \delta)^{k+1}) .
\]

Isolating \(\mu\) results in,

\[
\mu \leq -a\tilde{N}(1 - e^{-\lambda})(1 - \delta)^{-(k+1)} - 1 .
\]

Now we apply the hypothesis to obtain

\[
\mu \leq -a\tilde{N}(1 - e^{-\lambda}) \frac{1}{\ln \left(\frac{\lambda}{\gamma}\right)} = e^{-\lambda} - 1
\]

which proves that

\[
\mu \leq e^{-\lambda} - 1 .
\]  
(17)

A similar proof for the situation where \(x_{m_i-1-k}\) does not go to zero results in the inequality,

\[
\lambda \geq e^{-\mu} - 1 .
\]  
(18)

Finally, recall the case where \(x_{m_i-1-k} \to 0\) which resulted in \(\mu = 0\). Suppose that \(x_{m_i-1-k}\) does not go to zero. Then we have that

\[
\lambda \geq e^{-\mu} - 1 = 0 .
\]

But, \(\lambda \leq 0\) and hence \(\mu = \lambda = 0\) which concludes the proof of the theorem. Similarly if \(x_{m_i-1-k}\) converges to zero, we can show, as we did for \(\mu\), that \(\lambda = 0\) and again the entire theorem is proved. This completes Step 2.
Step 3: Using (17) and (18) we will show that \( \lambda = \mu = 0 \).

By (17) we have, e\(^{\mu} \leq e^{-\lambda - 1} \) or
\[
e^{1 - e^{-\lambda}} \leq e^{-\mu}.
\]
But by (18), e\(^{-\mu} \leq \lambda + 1 \) so
\[
e^{-e^{-\lambda}} \leq \frac{1}{e}\frac{1}{\lambda + 1}.
\]
(19)

Let \( f(\lambda) = e^{-e^{-\lambda}} \). By taking the first and second derivative of \( f \), we can see that \( f \) is a increasing function that is always concave up. Moreover, the tangent line at \( \lambda = 0 \) is \( y = 1/e(\lambda + 1) \). Therefore, \( 1/e(\lambda + 1) \leq f(\lambda) \) (see Fig. 6).

So Inequality (19) is satisfied only at \( \lambda = 0 \). Thus, \( \lambda = 0 \).

Since, \( \mu \leq e^{-\lambda} - 1 = 0 \) and \( \mu \geq 0 \) we have that \( \mu = 0 \). This shows that
\[
\lim_{n \rightarrow \infty} x_n = 0
\]
and the proof is complete. \( \square \)

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References