Class. Quantum Grav. **17** (2000) 2149–2152. Printed in the UK PII: S0264-9381(00)50010-0

# An invariant of null spinor fields

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Received 13 December 1999

**Abstract.** A differential invariant is found for null fields of arbitrary spin. For null gravitational fields, the invariant is identified as that of Bičác and Pravda; its derivation from the Bel–Robinson tensor is given; and a simple expression is found for its value in null perturbations of flat spacetimes.

PACS numbers: 0420, 0430

## 1. Introduction

All null solutions of Einstein's equations with twist-free rays have long been known [1–3]. For many years only one solution was known with twisting rays [4]; another has been found recently [5]. In addition, there is a good collection of approximate solutions [6–9]. There remains the problem of understanding these solutions and, in particular, of determining which of them are physically acceptable.

For this purpose, it is useful to study invariants of the Weyl tensor. In the null case, the algebraic invariants vanish identically. Bičác and Pravda [10] have investigated the differential invariants. They have shown that there are none of first order and exactly one of second order. Their computations are complicated. We shall derive their invariant rather more simply as a special case of an invariant formed from a null spinor field with $m$ indices. We begin with the case $m = 1$.

## 2. The deviation vector

For any spinor field, $\nu_A$, we define the *deviation vector*

$$\Theta_{AA'} = \nu^B \nabla_{AA'} \nu_B. \tag{1}$$

The associated self-dual null bivector and null vector field,

$$N_{AA'BB'} = \nu_A \nu_B \varepsilon_{A'B'}, \qquad n_{AA'} = \nu_A \overline{\nu}_{A'}, \tag{2}$$

satisfy

$$N_a{}^p \nabla_c N_{pb} = N_{ab} \Theta_c, \tag{3}$$

and

$$(\nabla_a n_p)(\nabla_b n^p) = -\left(\Theta_a \overline{\Theta}_b + \overline{\Theta}_a \Theta_b\right). \tag{4}$$

0264-9381/00/102149+04$30.00  

The optical scalars for a geodesic vector field $n^a$ can be derived algebraically from $\Theta_a$, $n_a$ and $N_{ab}$. Indeed, the following conditions are equivalent:

(i) $n_a$ is geodesic;
(ii) $n_a\Theta^a = 0$;
(iii) there exists a scalar $\rho$ such that $\Theta^a\overline{N}_{ab} = \rho n_b$;
(iv) there exists a scalar $\sigma$ such that $\Theta^a N_{ab} = \sigma n_b$.

When these conditions are satisfied, $\rho$ and $\sigma$ are respectively the complex expansion and shear of $n_a$; moreover

$$-\Theta_a\overline{\Theta}^a = \rho\overline{\rho} + \sigma\overline{\sigma}, \qquad -\Theta_a\Theta^a = 2\rho\sigma. \tag{5}$$

We are particularly interested in the case of geodesic and shear-free vector fields. There is then only one scalar to be formed by contracting $\Theta_a$, $\overline{\Theta}_a$, and $n_a$.

## 3. Null spinors

A symmetrical spinor is said to be null if its principal null directions are all coincident. An $m$-index null spinor may be written as

$$\varphi_{A_1\ldots A_m} = \psi\nu_{A_1}\ldots\nu_{A_m}. \tag{6}$$

As a first step in the construction of a differential invariant, we remark that for $r + s < m$,

$$(\nabla_{A_1A'_1}\ldots\nabla_{A_rA'_r}\varphi^{C_1\ldots C_m})\nabla_{B_1B'_1}\ldots\nabla_{B_sB'_s}\varphi_{C_1\ldots C_m} = 0. \tag{7}$$

Consequently, the tensor

$$D_{A_1A'_1\ldots A_mA'_m} = \frac{(-1)^r}{m!}(\nabla_{A_1A'_1}\ldots\nabla_{A_rA'_r}\varphi^{C_1\ldots C_m})\nabla_{A_{r+1}A'_{r+1}}\ldots\nabla_{A_mA'_m}\varphi_{C_1\ldots C_m} \tag{8}$$

is independent of $r$. Taking $r = 0$, one sees that

$$D_{a_1\ldots a_m} = \psi^2\Theta_{a_1}\ldots\Theta_{a_m}. \tag{9}$$

For $m = 2p$, these results can be expressed in tensorial form. Writing

$$F_{A_1A'_1\ldots A_mA'_m} = \varphi_{A_1\ldots A_m}\varepsilon_{A'_1A'_2}\ldots\varepsilon_{A'_{m-1}A'_m} + \text{c.c.} \tag{10}$$

we have

$$F_{a_1b_1\ldots a_pb_p} = \psi N_{a_1b_1}\ldots N_{a_pb_p} + \overline{\psi}\,\overline{N}_{a_1b_1}\ldots\overline{N}_{a_pb_p}; \tag{11}$$

and hence

$$D_{a_1\ldots a_{2p}} = \left(-\tfrac{1}{2}\right)^p(\nabla_{a_1}\ldots\nabla_{a_p}{}^+F^{b_1c_1\ldots b_pc_p})(\nabla_{a_{p+1}}\ldots\nabla_{a_{2p}}{}^+F_{b_1c_1\ldots b_pc_p}) \tag{12}$$

where, as usual,

$${}^+F^{b_1c_1\ldots b_pc_p} = \tfrac{1}{2}\left(F^{b_1c_1\ldots b_pc_p} - i{}^*F^{b_1c_1\ldots b_pc_p}\right) \tag{13}$$

the star denoting the Hodge dual, taken over any of the antisymmetric index pairs.

We can now construct the invariants

$$J = D_{a_1\ldots a_m}\overline{D}^{a_1\ldots a_m}, \tag{14}$$
$$J' = D_{a_1\ldots a_m}D^{a_1\ldots a_m}, \tag{15}$$
$$J'' = D_{a_1\ldots a_m}T^{a_1\ldots a_m} \tag{16}$$

where

$$T_{A_1A'_1\ldots A_mA'_m} = \varphi_{A_1\ldots A_m}\overline{\varphi}_{A'_1\ldots A'_m}. \tag{17}$$

We see that

$$J = (\psi\overline{\psi})^2 \left(\Theta_a\overline{\Theta}^a\right)^m, \tag{18}$$

$$J' = \psi^4 \left(\Theta_a\Theta^a\right)^m, \tag{19}$$

$$J'' = \psi^3\overline{\psi} \left(\Theta_a n^a\right)^m \tag{20}$$

For $m = 4$, taking $F_{abcd}$ to be the Weyl tensor, we see that $J$ is, to a numerical factor, the invariant of Bičác and Pravda. When Einstein's equations (with cosmological constant) are satisfied, $J'$ and $J''$ vanish, because $n_a$ is geodesic and shear-free.

## 4. Alternative approach

$J$ is invariant under the transformation

$$\varphi_{A_1\ldots A_m} \longmapsto e^{i\theta}\varphi_{A_1\ldots A_m} \tag{21}$$

and $T^{a_1\ldots a_m}$ determines the spinor up to this transformation. $J$ is therefore an invariant of $T^{a_1\ldots a_m}$. For $m = 2$ and $m = 4$ one obtains

$$(\nabla_a\nabla_b T_{cd})\nabla^a\nabla^b T^{cd} = (\psi\overline{\psi})^2 \left[\left(4\Theta_a\overline{\Theta}^a\right)^2 + 8\left|\Theta_a\Theta^a\right|^2\right] \tag{22}$$

and

$$(\nabla_a\nabla_b\nabla_c\nabla_d P_{rstu})\nabla^a\nabla^b\nabla^c\nabla^d P^{rstu}$$
$$= (6\psi\overline{\psi})^2 \left[\left(4\Theta_a\overline{\Theta}^a\right)^4 + 753\left(\Theta_a\overline{\Theta}^a\right)^2\left|\Theta_a\Theta^a\right|^2 + 96\left|\Theta_a\Theta^a\right|^4\right] \tag{23}$$

which reduce to multiples of $J$ when $n_a$ is geodesic and shear-free.

## 5. Application

For null perturbations of flat space, we can use this invariant to test for singularities without actually solving the perturbed equations. In standard notation [11], with coordinates $r$, $u$, $\zeta$, $\overline{\zeta}$, a null gravitational field is determined by functions $P$ and $L$ of $u$, $\zeta$ and $\overline{\zeta}$. Suppose that $\underset{0}{P}$ and $\underset{0}{L}$ are the lowest approximations to these functions; $\underset{0}{\rho}$ and $\underset{0}{I}$ the complex expansion and the function $I$ calculated from them; and choose coordinates in which $\underset{0}{I} = 0$. Let $\tau(u, \zeta, \overline{\zeta})$ be any solution of

$$\overline{\partial}\tau \equiv \tau_{\overline{\zeta}} - \underset{0}{\overline{L}}\tau_u = 0, \qquad \tau_u \neq 0. \tag{24}$$

Then in the lowest non-trivial approximation

$$J = \left|\underset{0}{\rho}^3\ \underset{0}{P}^2\tau_u f(\zeta, \tau)\right|^4 \tag{25}$$

for some function $f(\zeta, \tau)$.

As an example, we consider a simple model of radiation from a source in arbitrary motion. In the lowest approximation, the source appears as a time-like curve in Minkowski space and the direction of radiation as the future-directed null lines emanating from it. We then obtain

$$\underset{0}{P} = a + \tfrac{1}{2}a^{-1}(b+\zeta)(\overline{b}+\overline{\zeta}), \qquad \underset{0}{L} = 0, \tag{26}$$

where $a(> 0)$, $b$, $\overline{b}$ are functions of $u$. Then

$$J = \left|r^{-3}\ \underset{0}{P}^2 f(\zeta, \tau)\right|^4. \tag{27}$$

It is an immediate consequence of Liouville's theorem that $J$ cannot be bounded for all $\zeta$. There is, therefore, a directional singularity in the radiation field, as was first shown by Stephani [7] for radiation from an unaccelerated source.

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