

LETTER

Note on Invariants of the Weyl Tensor

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Received April 30, 2003

Algebraically special gravitational fields are described using algebraic and differential invariants of the Weyl tensor. A type III invariant is also given and calculated for Robinson-Trautman spaces.

KEY WORDS: Invariants; algebraic classification of the Weyl tensor.

1. INTRODUCTION

It is well known (see [1] and [2]) that there are exactly two algebraic invariants to be constructed from the Weyl tensor, namely

$$I = \frac{1}{4} {}^+C_{abcd} {}^+C^{abcd}, \tag{1}$$

$$J = \frac{1}{8} {}^+C_{abcd} {}^+C_{ef}{}^{cd} {}^+C^{efab}, \tag{2}$$

where ${}^+C$ is the self-dual part of the Weyl tensor. Provided that $I^3 - 6J^2 \neq 0$, they determine the intensity of the gravitational field or, more precisely, they determine the Weyl tensor at any point up to a local rotation of frame. These invariants were also shown in [2] to be related with the eigenspinors of the Weyl spinor as well as with the cross ratio of the gravitational principal null directions. Specifically, if χ

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is the cross ratio of any four null directions then

$$I^3 \left[(\chi + 1)(\chi - 2) \left(\chi - \frac{1}{2} \right) \right]^2 = 6J^2 [(\chi + \omega)(\chi + \omega^2)]^3 \tag{3}$$

where $\omega = e^{\frac{2\pi i}{3}}$ is the cube root of unity. In particular $I^3 = 6J^2 \neq 0$ if and only if $\chi \in \{0, 1, \infty\}$ if and only if the space-time is of type (2, 1, 1) or (2, 2).

It is apparent that the two algebraic invariants can provide only a partial classification of the degree of degeneracy of the Weyl tensor. As shown above, $I^3 = 6J^2 \neq 0$ is true for both (2, 2) and (2, 1, 1) cases while $I = J = 0$ covers all types (3, 1), (4) and flat space. For a more precise classification one has to use differential invariants.

All differential invariants vanish for flat space-time. The same is true for types (4) and (3, 1) space-times built around a nonexpanding congruence of null rays (see [3] and [4]). However, for type N expanding space-times, Bicak and Pravda showed in [3] that there exist exactly one nonzero differential invariant of the second order. That invariant was shown to be useful in analyzing Finley, Plebanski and Przanowski twisting, type N approximate solutions obtained in [5]; they showed that those solutions contain singularities at large distances and hence cannot describe radiation fields outside bounded sources. An alternate derivation of that invariant has been given in [6]. Pravda found another invariant for type (3, 1) in [4].

In this paper we show that both an invariant for the case (2, 1, 1) and Pravda’s invariant can be derived from Bicak and Pravda type (4) invariant. A complete classification of the degeneracies of the Weyl tensor is shown to be possible using the algebraic and the differential invariants mentioned above. A new second order differential invariant is proposed and its value is calculated for the Robinson-Trautman solutions.

2. CLASSIFICATION

For any F_{abcd} with symmetries similar to the ones of ${}^+C$ we define

$$J_F = F_{abcd;rs} F^{abcd} {}_{;tu} \bar{F}{}_{efgh;rs} \bar{F}{}^{efgh;tu} \tag{4}$$

remark that for any null field F , J_F is the invariant J in [6]. We are particularly interested in J_A , J_B and J_+C where

$$A_{abcd} = IB_{abcd} - J^+C_{abcd} \tag{5}$$

$$B_{cd}^{ab} = \frac{1}{2} {}^+C_{rs}^{ab} {}^+C_{cd}^{sr} - \frac{1}{3} I^+ \delta_{cd}^{ab} \tag{6}$$

where $\delta_{abcd} = \frac{1}{2} (g_{ad}g_{bc} - g_{ac}g_{bd} - i\eta_{abcd})$, η being the Levi-Civita tensor. Notice that when $I^3 = 6J^2$ the tensor A is null ($A_{abcd} = 6\Psi_2^2(3\Psi_2\Psi_4 - \Psi_3^2)N_{ab}N_{cd}$) in

the case (2, 1, 1) and it vanishes in the more degenerate cases (see [1]); for $I = J = 0$ the tensor B is null ($B_{abcd} = -4\Psi_3^2 N_{ab} N_{cd}$) in the (3, 1) case and zero otherwise. Moreover

$$J_A = |96\Psi_2^2 (3\Psi_2\Psi_4 - \Psi_3^2) \rho^2|^4, \tag{7}$$

$$J_B = |8\Psi_3\rho|^8. \tag{8}$$

In conclusion, for space-times admitting an expanding congruence we have the following classification:

- $-I^3 \neq 6J^2, I \neq 0, J \neq 0 : (1, 1, 1, 1);$
- $-I^3 = 6J^2 \neq 0, J_A \neq 0 : (2, 1, 1);$
- $-I^3 = 6J^2 \neq 0, J_A = 0 : (2, 2);$
- $-I = J = 0, J_B \neq 0 : (3, 1);$
- $-I = J = 0, J_B = 0, J_{+C} \neq 0 : (4);$
- $-I = J = 0, J_B = 0, J_{+C} = 0 : (-).$

3. FURTHER REMARKS ON THE (3, 1) CASE

For $I = J = 0$ case we can alternatively use the first order invariant obtained in [4]

$$J_P = C^{abcd:e} C_{amcn:e} C^{lmrn:s} C_{lbrd:s} \tag{9}$$

to distinct (3, 1) case from more degenerate ones.

We did not investigate systematically invariants of second order but we mention that if

$$D_{rst} = {}^+C_{abcd;r} {}^+C^{abcd}{}_{;st} \tag{10}$$

then

$$D = D_{[rst]} \overline{D}^{[rst]} \tag{11}$$

has the following expression for a (3, 1) Robinson-Trautman solution with $P = P(\sigma, \xi, \eta)$:

$$D = \frac{36p^4}{r^{14}} (K_\xi^2 + K_\eta^2) \left[\frac{1}{8} (K_\xi^2 + K_\eta^2) K + p (K_{\xi\eta}^2 - K_{\xi\xi} K_{\eta\eta}) \right] \tag{12}$$

$$+ 9 \frac{p^4}{r^{13}} \left[(K_\xi^2 + K_\eta^2)^2 \right], \sigma. \tag{13}$$

Remark that for (3, 1) and (4) cases, the geometry of each light cone is independent of the one of its neighbors; and both J_P and J_B depend only on the geometry of each individual light cone. However, the invariant D also depends on the rate of change of the geometry from one light cone to another.

ACKNOWLEDGEMENT

The author is grateful to Ivor Robinson for his insights and comments.

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