LETTER

Note on Invariants of the Weyl Tensor

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Algebraically special gravitational fields are described using algebraic and differential invariants of the Weyl tensor. A type III invariant is also given and calculated for Robinson-Trautman spaces.

KEY WORDS: Invariants; algebraic classification of the Weyl tensor.

1. INTRODUCTION

It is well known (see [1] and [2]) that there are exactly two algebraic invariants to be constructed from the Weyl tensor, namely

\[ I = \frac{1}{4} C_{abcd}^+ C^{abcd}, \]
\[ J = \frac{1}{8} C_{abcd}^+ C_{\varepsilon \delta \beta \alpha}^+ C_{\varepsilon \delta \beta \alpha}, \]

where \( C^+ \) is the self-dual part of the Weyl tensor. Provided that \( I^3 - 6J^2 \neq 0 \), they determine the intensity of the gravitational field or, more precisely, they determine the Weyl tensor at any point up to a local rotation of frame. These invariants were also shown in [2] to be related with the eigenspinors of the Weyl spinor as well as with the cross ratio of the gravitational principal null directions. Specifically, if \( \chi \)

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is the cross ratio of any four null directions then
\[ I^3 \left[ (\chi + 1)(\chi - 2) \left( \chi - \frac{1}{2} \right) \right]^2 = 6J^2[(\chi + \omega)(\chi + \omega^2)]^3 \]  
(3)
where \( \omega = e^{\frac{i\pi}{3}} \) is the cube root of unity. In particular \( I^3 = 6J^2 \neq 0 \) if and only if \( \chi \in \{0, 1, \infty\} \) if and only if the space-time is of type (2, 1, 1) or (2, 2).

It is apparent that the two algebraic invariants can provide only a partial classification of the degree of degeneracy of the Weyl tensor. As shown above, \( I^3 = 6J^2 \neq 0 \) is true for both (2, 2) and (2, 1, 1) cases while \( I = J = 0 \) covers all types (3, 1), (4) and flat space. For a more precise classification one has to use differential invariants.

All differential invariants vanish for flat space-time. The same is true for types (4) and (3, 1) space-times built around a nonexpanding congruence of null rays (see [3] and [4]). However, for type N expanding space-times, Bicak and Pravda showed in [3] that there exist exactly one nonzero differential invariant of the second order. That invariant was shown to be useful in analyzing Finley, Plebanski and Przeganowski twisting, type N approximate solutions obtained in [5]; they showed that those solutions contain singularities at large distances and hence cannot describe radiation fields outside bounded sources. An alternate derivation of that invariant has been given in [6]. Pravda found another invariant for type (3, 1) in [4].

In this paper we show that both an invariant for the case (2, 1, 1) and Pravda’s invariant can be derived from Bicak and Pravda type (4) invariant. A complete classification of the degeneracies of the Weyl tensor is shown to be possible using the algebraic and the differential invariants mentioned above. A new second order differential invariant is proposed and its value is calculated for the Robinson-Trautman solutions.

2. CLASSIFICATION

For any \( F_{abcd} \) with symmetries similar to the ones of \( ^+C \) we define
\[ J_F = F_{abcd;ri}F^*_{abcd;st}F^*_{efgh;rs}F^*_{efgh;tu} \]  
(4)
remark that for any null field \( F \), \( J_F \) is the invariant \( J \) in [6]. We are particularly interested in \( J_A, J_B \) and \( J_C \) where
\[ A_{abcd} = IB_{abcd} - J^+C_{abcd} \]  
(5)
\[ B_{cd} = \frac{1}{2} C_{cd} + C_{sr} - \frac{1}{3} I^{+}\delta_{cd} \]  
(6)
where \( \delta_{abcd} = \frac{1}{2} (g_{ad}g_{bc} - g_{ac}g_{bd} - \eta_{abcd}) \), \( \eta \) being the Levi-Civita tensor. Notice that when \( I^3 = 6J^2 \) the tensor \( A \) is null \( (A_{abcd} = 6\Psi_2^2 (3\Psi_2 \Psi_4 - \Psi_3^2) N_{ab} N_{cd}) \) in
the case (2, 1, 1) and it vanishes in the more degenerate cases (see [1]); for $I = J = 0$ the tensor $B$ is null ($B_{abcd} = -4\Psi_2^2 N_{ab} N_{cd}$) in the (3, 1) case and zero otherwise. Moreover

$$J_A = |96\Psi_2^2 (3\Psi_2\Psi_4 - \Psi_5^2) \rho^2|^4,$$

$$J_B = |8\Psi_5 \rho|^8.$$  \hfill (7)

In conclusion, for space-times admitting an expanding congruence we have the following classification:

- $I^3 \neq 6J^2$, $I \neq 0$, $J \neq 0$ : (1, 1, 1, 1);
- $I^3 = 6J^2 \neq 0$, $J_A \neq 0$ : (2, 1, 1);
- $I^3 = 6J^2 \neq 0$, $J_A = 0$ : (2, 2);
- $I = J = 0$, $J_B \neq 0$ : (3, 1);
- $I = J = 0$, $J_B = 0$, $J_C \neq 0$ : (4);
- $I = J = 0$, $J_B = 0$, $J_C = 0$ : (\(-\)).

3. FURTHER REMARKS ON THE (3, 1) CASE

For $I = J = 0$ case we can alternatively use the first order invariant obtained in [4]

$$J_P = C^{abcd} C_{amcn} C^{lmns} C_{brdx}$$  \hfill (9)

to distinct (3, 1) case from more degenerate ones.

We did not investigate systematically invariants of second order but we mention that if

$$D_{rst} = + C_{abcd} + C_{abcd \; ,rst}$$  \hfill (10)

then

$$D = D_{\{\alpha\beta\}} D^{\{\alpha\beta\}}$$  \hfill (11)

has the following expression for a (3, 1) Robinson-Trautman solution with $P = P(\sigma, \xi, \eta)$:

$$D = \frac{36\rho^4}{r^{14}} (K_2^2 + K_4^2) \left[ \frac{1}{8} (K_2^2 + K_4^2) K + p (K_2^2 - K_2^2 K_4^2) \right]$$  \hfill (12)

$$+ 9\frac{\rho^4}{r^{13}} \left[ (K_2^2 + K_4^2)^2 \right]_{,\sigma}.$$  \hfill (13)

Remark that for (3, 1) and (4) cases, the geometry of each light cone is independent of the one of its neighbors; and both $J_P$ and $J_B$ depend only on the geometry of each individual light cone. However, the invariant $D$ also depends on the rate of change of the geometry from one light cone to another.
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