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PURE GRAVITATIONAL RADIATION WITH TWISTING RAYS

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To my wife

# PURE GRAVITATIONAL RADIATION WITH TWISTING RAYS

by

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DISSERTATION

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Chapter two makes the subject of the paper "Classification of gravitational fields by means of invariants" which, at the time this thesis was written, was submitted to *General Relativity and Gravitation.* The expression 3.32 from Chapter three for the second order invariant calculated for a Robinson-Trautman type III space-time was also included in that paper.

A modified version of Chapter four was posted as a preprint gr-qc/0105064 at http://xxx.lanl.gov under the title "Regular Type III and Type N Approximate Solutions".

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PURE GRAVITATIONAL RADIATION WITH TWISTING RAYS

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Chapter one presents a few topics in General Relativity with emphasis on algebraically special solution especially those of type III and N. The only known type III solutions (Robinson-Trautman and its "twisted" up counterpart) are briefly described.

Chapter two gives a classification of gravitational fields in terms of algebraic and differential invariants. Algebraic invariants divide the algebraically special fields in three classes; members of these classes are distinguished by specific differential invariants.

Chapter three concentrates on type III space-time, specifically on their description using differential invariants. An invariant of order one is presented and calculated for the available exact solutions. It shows that both of them contain directional singularities. An invariant of order two is also discovered and calculated for Robinson-Trautman solutions.

In Chapter four new type III and type N approximate solutions which are regular in the linear approximation are shown to exist. For that, we use complex transformations on self-dual Robinson-Trautman metrics rather then the classical approach. The regularity criterion is the boundedness and vanishing at infinity of a scalar obtained by saturating the Bel-Robinson tensor of the first approximation by a time-like vector which is constant with respect to the zeroth approximation.

The solutions obtained in Chapter four are continued in Chapter five to the second order. The classical method is employed this time since the previously used theory is no longer Lorentz invariant. We obtain a partial solution for which the invariant of order one described earlier is regular. We also give expressions for the coefficients of the Weyl tensor and show their regularity.

# Conventions used in the text

### Metric and tetrads

Signature of space-time metric: (+ - - -)

Complex null tetrad: $n_a$, $l_a$, $m_a$, $\overline{m}_a$; $g_{ab} = n_a l_b + l_a n_b - m_a \overline{m}_b - \overline{m}_a m_b$

### Bivectors

Dual bivector: $X^*_{ab} = \frac{1}{2}\sqrt{-g}\varepsilon_{abcd} X^{dc}$

Self-dual part: ${}^+X_{ab} = \frac{1}{2}\left(X_{ab} + iX^*_{ab}\right)$; ${}^+X^*_{ab} = -i{}^+X_{ab}$

Anti-self-dual part: ${}^-X_{ab} = \frac{1}{2}\left(X_{ab} - iX^*_{ab}\right)$; ${}^-X^*_{ab} = i{}^-X_{ab}$

### Weyl tensor

Self-dual part of the Weyl tensor: ${}^+C_{abcd} = \frac{1}{2}\left(R_{abcd} + iR^*_{abcd}\right)$

Self-dual bivector basis: $N_{ab} = n_a m_b - m_a n_b$,

$$M_{ab} = n_a l_b - l_a n_b + m_a \overline{m}_b - \overline{m}_a m_b,$$

$$L_{ab} = \overline{m}_a l_b - l_a \overline{m}_b$$

Weyl coefficients: 
$$\begin{aligned} {}^+C_{abcd} = {} & \Psi_0 L_{ab} L_{cd} + \Psi_1 \left(M_{ab} L_{cd} + L_{ab} M_{cd}\right) \\ & + \Psi_2 \left(N_{ab} L_{cd} + L_{ab} N_{cd} + M_{ab} M_{cd}\right) \\ & + \Psi_3 \left(N_{ab} M_{cd} + M_{ab} N_{cd}\right) + \Psi_4 N_{ab} N_{cd} \end{aligned}$$

# CHAPTER 1

# SOME TOPICS IN GENERAL RELATIVITY

This chapter presents a few topics in General Relativity which are going to be of interest later on.

## 1.1 Null vector fields

In a four dimensional Riemannian space, vector fields are often characterized using properties of their first covariant derivatives and of some special scalars obtained from these derivatives. Although scalars can be defined for all types of vector fields (time-like, space-like and null), we concentrate on null ones since they are going to be of most interest in future sections.

A vector field $n$ in a Riemannian space endowed with the metric $g$ is called *time-like, space-like* or *null* if it satisfies $g_{ab}n^a n^b < 0$, $g_{ab}n^a n^b > 0$ or $g_{ab}n^a n^b = 0$ respectively; $n$ is called *geodesic* if $l_{a;b}l^b \propto l_a$. Moreover, if the vector field is *affinely parametrized* the relation becomes

$$l_{a;b}l^b = 0. \tag{1.1}$$

Physically, a geodesic null vector field represents a vector field tangent to a null congruence of optical rays.

For an affinely normalized geodesic congruence of null rays, Sachs introduced (in [28]) and studied (in [28] and [29]) three invariant quantities, called *optical scalars,* in terms of the vector field $n$ tangent to the rays of the congruence. They are *expansion* (or divergence), *twist* (or rotation) and *shear* and have the respective expressions

$$\theta = \frac{1}{2} n^a{}_{;a}, \tag{1.2}$$

$$\omega^2 = \frac{1}{2} n_{[a;b]} n^{a;b}, \tag{1.3}$$

$$\sigma\overline{\sigma} = \frac{1}{2} n_{(a;b)} n^{a;b} - \frac{1}{4} (n^a{}_{;a})^2. \tag{1.4}$$

Alternative formulae in terms of a complex null tetrad will be given in the next section.

Obviously a vector field tangent to such congruence of null rays can be characterized be means of these optical scalars; the importance of such characterization will be apparent later on. At this point we only mention two of the most important uses of it. First, if the vector field is a preferred one with respect to a particular solution (a repeated null direction for example), then a characterization of it is actually a characterization of that solution. Second, existence of a vector field with special properties imposes conditions on the metric itself (for example the Robinson -Trautman metrics are exactly the ones built around a geodesic, shear-free non-twisting but expanding congruence).

A very simple but useful relation is obtained when evaluating the expression

$$n_{a;bc} - n_{a;cb} = R^d{}_{abc} n_d \tag{1.5}$$

where $R_{abcd}$ is the curvature tensor. Contracting and using equations 1.2, 1.3 and 1.4 one obtains

$$\theta_{,a} n^a - \omega^2 + \theta^2 + \sigma\overline{\sigma} = \frac{1}{2} R_{ab} n^a n^b \tag{1.6}$$

where $R_{ab}$ is the Ricci tensor. In an empty space ($R_{ab} = 0$), the previous equation becomes

$$\theta_{,a} n^a - \omega^2 + \theta^2 + \sigma\overline{\sigma} = 0. \tag{1.7}$$

This last expression shows that there exist no empty space which admits a non-expanding, shear-free and twisting congruence.

## 1.2 Complex null tetrads

The standard way of treating general relativity problems before 1960 used to be the consideration of Einstein field equations in a local coordinate system adapted to that specific problem. However, in more recent years it has become advantageous to take a different approach, namely to choose a suitable tetrad basis of four linearly independent vector fields and project the field equations along these vector fields. This approach is called the *tetrad formalism*.

In 1962 Newman and Penrose improved this theory by making a special choice of the vectors in the basis (they were all chosen to be orthogonal null vector fields, two of them real and the other two complex conjugated and such that the scalar product of the two real ones equals 1 and of the two complex ones equals -1). This new approach, called the Newman-Penrose formalism, proves to be very useful in investigating algebraically special gravitational fields. This section presents only the features that are going to be used later on; for a full description of the theory as well applications see [12], [7], [10], [34].

Let $n_a$, $l_a$, $m_a$, $\overline{m}_a$ as described above, i.e. such that $l_a n^a = 1$, $m_a\ \overline{m}^a = -1$ and all others scalar products vanish. The components of the metric with respect to this null tetrad are

$$g_{ab} = n_a l_b + l_a n_b - m_a \overline{m}_b - \overline{m}_a m_b. \tag{1.8}$$

In terms of this tetrad, one can define the so-called *spin coefficients*, 12 independent complex linear combinations of the connection coefficients (see [12]). They are

$$-\kappa = n_{a;b} m^a n^b, \tag{1.9}$$

$$-\rho = n_{a;b} m^a \overline{m}^b, \tag{1.10}$$

$$-\sigma = n_{a;b} m^a m^b \tag{1.11}$$

$$-\tau = n_{a;b} m^a l^b, \tag{1.12}$$

$$\nu = l_{a;b} \overline{m}^a l^b, \tag{1.13}$$

$$\mu = l_{a;b} \overline{m}^a m^b, \tag{1.14}$$

$$\lambda = l_{a;b} \overline{m}^a \overline{m}^b, \tag{1.15}$$

$$\pi = l_{a;b} \overline{m}^a n^b, \tag{1.16}$$

$$-\varepsilon = \frac{1}{2} \left( n_{a;b} l^a n^b - m_{a;b} \overline{m}^a n^b \right), \tag{1.17}$$

$$-\beta = \frac{1}{2} \left( n_{a;b} l^a m^b - m_{a;b} \overline{m}^a m^b \right), \tag{1.18}$$

$$\gamma = \frac{1}{2} \left( l_{a;b} n^a l^b - \overline{m}_{a;b} m^a l^b \right), \tag{1.19}$$

$$\alpha = \frac{1}{2} \left( l_{a;b} n^a \overline{m}^b - \overline{m}_{a;b} m^a \overline{m}^b \right). \tag{1.20}$$

The spin coefficient $\sigma$ has already been introduced by 1.4. Using these and directional derivatives along the tetrad, one can write a complete set of equations from which exact solutions of Einstein's field equations may be found (see [12]) and, although the number of equations is high, all equations are linear. Another advantage of using spin coefficients is that the algebraic properties of the tetrad can be easily read if we know them. For example, the quantities $\kappa$, $\rho$, $\sigma$ are specific to the null congruence defined by the vector $n$ as follows:

- $\kappa$ is a measure of the congruence being geodesic or not; in fact the congruence is geodesic if and only if $\kappa = 0$;

- $Re\rho$ called *dilation* (or expansion) is a measure of the rate of contraction of the bundles of rays;

-$Im\rho$ called *twist* measures the rotation of the rays; the connection between this $\rho$ and the previously defined twist 1.3 and expansion 1.2 is

$$\rho = -(\theta + i\omega). \tag{1.21}$$

-$\sigma$ is called the shear since $|\sigma|$ measures the degree of shearing and $\frac{1}{2}\arg\sigma$defines the angle of shearing.

For a complete discussion of the spin coefficients and their interpretations see [15] and [16].

Next, one can define three bivectors

$$N_{ab} = n_a m_b - m_a n_b, \tag{1.22}$$

$$M_{ab} = n_a l_b - l_a n_b + \overline{m}_a m_b - m_a \overline{m}_b, \tag{1.23}$$

$$L_{ab} = \overline{m}_a l_b - l_a \overline{m}_b. \tag{1.24}$$

It can easily be checked that the bivectors and their complex conjugates satisfy

$$N_{ab}L^{ab} = 2, \tag{1.25}$$

$$M_{ab}M^{ab} = -4, \tag{1.26}$$

all other products of this type being zero, as well as

$$N_{ac}N^{c}{}_{b} = 0, \quad N_{ac}\overline{N}^{c}{}_{b} = 0 \tag{1.27}$$

$$N_{ac}M^{c}{}_{b} = -N_{ab}, \quad N_{ac}\overline{M}^{c}{}_{b} = m_a n_b + n_a m_b \tag{1.28}$$

$$M_{ac}\overline{M}^{c}{}_{b} = l_a n_b + n_a l_b + m_a \overline{m}_b + \overline{m}_a m_b. \tag{1.29}$$

We call a bivector $N$ self-dual if $N^* = -iN$ where the star represents the dual taken with respect to any pair of antisymmetric indices

$$W_{\cdot\cdot}{}^{*}_{ab}{}_{\cdots} = \frac{1}{2}(-|g|)^{1/2}\epsilon_{abcd}W_{\cdots}{}^{dc}{}_{\cdots} \quad , \tag{1.30}$$

$\epsilon_{abcd}$ representing the Levi-Civita completely antisymmetric tensor defined by $\epsilon_{1234} = -1$ and $|g|$ represents the determinant of $g_{ab}$. One can easily see that the bivectors defined by 1.22, 1.23 and 1.24 are self dual hence they form a basis in the self-dual bivectors space. The gradient of any one of them is still self-dual hence it can be expressed as a linear combination with bivectors in any basis. Using the previously calculated one and two products between bivectors, one finds (this approach was first used by Robinson and Schild in [21] to generalize Goldberg-Sachs theorem)

$$N_{ab;c} = N_{ab}U_c + M_{ab}\Theta_c, \tag{1.31}$$

$$M_{ab;c} = 2N_{ab}X_c + 2L_{ab}\Theta_c, \tag{1.32}$$

$$L_{ab;c} = M_{ab}X_c - L_{ab}U_c, \tag{1.33}$$

where $U$, $\Theta$, $X$ are vector fields which satisfy

$$\Theta_c = -\frac{1}{4}M^{ab}N_{ab;c} = n^a m_{a;c} \tag{1.34}$$

$$U_c = -\frac{1}{2}N^{ab}L_{ab;c} = l^a n_{a;c} + m^a \overline{m}_{a;c} \tag{1.35}$$

$$X_c = \frac{1}{4}L^{ab}M_{ab;c} = \overline{m}^a l_{a;c}. \tag{1.36}$$

It is interesting to know also the expressions of the *deviation vector* $\Theta_c$ (whose full properties can be found in [4]) and its analogue $U_c$ in terms of spin coefficients. They are

$$\Theta_c = -\tau n_c - \kappa l_c + \rho m_c + \sigma \overline{m}_c \tag{1.37}$$

$$U_c = 2(\gamma n_c + \varepsilon l_c - \alpha m_c - \beta \overline{m}_c). \tag{1.38}$$

where $\tau$, $\kappa$, $\rho$, $\sigma$, $\gamma$, $\varepsilon$, $\alpha$, $\beta$ are defined above.

The covariant derivative of a geodesic, affinely parametrized null vector field can be decomposed using the spin coefficients as

$$n_{a;b} = 2Re\left[(\theta + i\omega)\,\overline{m}_a m_b - \sigma \overline{m}_a \overline{m}_b\right] + n_a v_b + w_a n_b \tag{1.39}$$

where $v$ and $w$ are null vector fields such that $v_a n^a = w_a n^a = 0$.

## 1.3 The Weyl tensor

The curvature tensor can be uniquely decomposed as

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd} \tag{1.40}$$

where

$$E_{abcd} = \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}) \tag{1.41}$$

$$G_{abcd} = \frac{R}{12}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{1.42}$$

$$S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}, \quad R = R^a{}_a. \tag{1.43}$$

This decomposition defines the completely traceless Weyl tensor $C_{abcd}$. Moreover, all the tensors that appear in expression 1.40 have the same type of symmetries as the curvature tensor, i.e. if $X_{abcd}$ is any of the tensors $R$, $C$, $E$, $G$ in the equation 1.40 then

$$X_{abcd} = -X_{abdc} = -X_{bacd} = X_{cdab}, \tag{1.44}$$

$$X^a{}_{[bcd]} = 0 \tag{1.45}$$

For any of the afore mentioned tensors we can introduce a left dual and a right dual, depending on which pair of antisymmetric indices we apply the contraction:

$${}^*X_{abcd} = \frac{1}{2}(-|g|)^{1/2}\epsilon_{abef}X^{fe}{}_{cd} \tag{1.46}$$

$$X^*{}_{abcd} = \frac{1}{2}(-|g|)^{1/2}\epsilon_{cdef}X_{ab}{}^{fe}. \tag{1.47}$$

It turns out that for these particular tensors we have

$${}^*C_{abcd} = C^*{}_{abcd} \tag{1.48}$$

$${}^*E_{abcd} = -E^*{}_{abcd} \tag{1.49}$$

$${}^*G_{abcd} = G^*{}_{abcd}. \tag{1.50}$$

Without any ambiguity, we can introduce the self-dual part and the anti-self-dual part of the Weyl tensor as

$${}^+C_{abcd} = \frac{1}{2}\left(C_{abcd} + i^*C_{abcd}\right) \tag{1.51}$$

$${}^-C_{abcd} = \frac{1}{2}\left(C_{abcd} - i^*C_{abcd}\right) \tag{1.52}$$

respectively. It is easy to check that ${}^+C$ is a self-dual tensor while ${}^-C$ is an anti-self-dual one and

$$C_{abcd} =^+ C_{abcd} +^- C_{abcd}. \tag{1.53}$$

The complex space of all self-dual four-tensors with the symmetries similar to those of the Weyl tensor is 5 dimensional and a basis is given by $N_{ab}N_{cd}$, $N_{ab}M_{cd} + M_{ab}N_{cd}$, $N_{ab}L_{cd} + L_{ab}N_{cd} + M_{ab}M_{cd}$, $M_{ab}L_{cd} + L_{ab}M_{cd}$, $L_{ab}L_{cd}$ where $N$, $M$, $L$ are the bivectors defined in 1.22, 1.23 and 1.24 respectively. In this basis, the self-dual-part of the Weyl tensor has the following components

$$\begin{aligned} {}^{+}C_{abcd} = {} & \Psi_0 L_{ab}L_{cd} + \Psi_1 \left(M_{ab}L_{cd} + L_{ab}M_{cd}\right) + \Psi_2 \left(N_{ab}L_{cd} + L_{ab}N_{cd} + M_{ab}M_{cd}\right) \\ & + \Psi_3 \left(N_{ab}M_{cd} + M_{ab}N_{cd}\right) + \Psi_4 N_{ab}N_{cd} \end{aligned} \tag{1.54}$$

where $\Psi_i$ with $i = 0...4$ are given by

$$\Psi_0 = C_{abcd} n^a m^b n^c m^d, \tag{1.55}$$

$$\Psi_1 = C_{abcd} n^a l^b n^c m^d, \tag{1.56}$$

$$\Psi_2 = \frac{1}{2} C_{abcd} n^a l^b \left(n^c l^d - m^c \overline{m}^d\right), \tag{1.57}$$

$$\Psi_3 = C_{abcd} l^a n^b l^c \overline{m}^d, \tag{1.58}$$

$$\Psi_4 = C_{abcd} l^a \overline{m}^b l^c \overline{m}^d. \tag{1.59}$$

Szekeres gave in [33] the following interpretation for the various terms appearing in the expression of ${}^{+}C$ : in this particular frame of reference, $\Psi_4$ term represents a transverse wave in the $n$ direction, $\Psi_3$ term represents a longitudinal wave component and $\Psi_2$ a 'Coulomb" component; $\Psi_0$ and $\Psi_1$terms represent transverse and longitudinal wave components in the $l$ direction.

## 1.4 Algebraically special Weyl Tensor and Petrov Types

Consider the following bivector equations

$$X = {}^{+}X, \qquad X_{pq}X^{qp} = 0 \tag{1.60}$$

as well as

$$X_{pq}{}^{+}C^{qp}{}_{rs}X^{sr} = 0 \tag{1.61}$$

$$X_{pq}{}^{+}C^{qp}{}_{s[a}X_{b]}{}^{s} = 0 \tag{1.62}$$

$$X_{pq}{}^{+}C^{qp}{}_{ab} = 0 \tag{1.63}$$

$$^{+}C_{abs[c}X_{d]}{}^{s} = 0. \tag{1.64}$$

It is well known that the system of equations formed with 1.60 and 1.61 always has four solutions called *principal null bivectors* (or PNB) of the Weyl tensor. They may be distinct or coincident in various ways; if the Weyl tensor admits a repeated PNB then it is called *algebraically special*. The conditions that a solution is a repeated PNB involve equations 1.62, 1.63 and 1.64 and describe the so called Petrov types of the Weyl tensor as follows.

Let $N$ be a solution of equation 1.60. Then

-if $N$ satisfies equation 1.61 then it is at least PNB. The Weyl tensor is of Petrov type (1,1,1,1) or more degenerate;

-if $N$ satisfies equation 1.62 then it is at least a double PNB. The Weyl tensor is of Petrov type (2,1,1) or more degenerate.

-if $N$ satisfies equation 1.63 then it is at least a triple PNB. The Weyl tensor is of Petrov type (3,1) or more degenerate.

-if $N$ satisfies equation 1.64 then it is a quadruple PNB. The Weyl tensor is of Petrov type (4) or more degenerate (here "more degenerate" refers to the only possible degeneracy left, namely flat space attained for a zero Weyl tensor).

Let $M$ be another solution of equation 1.60. If $M$ and $N$ both satisfy equation 1.62 then the Weyl tensor is of type (2,2).

Note that this characterization does not depend on any particular null tetrad. However, if we take $N$ to be defined in terms of an adapted null tetrad by equation 1.22 we can translate all the criteria for the Petrov types in terms of Weyl coefficients $\Psi_i$ with $i = 0..4$ as follows:

- type (1,1,1,1) if $\Psi_0 = 0, \Psi_1 \neq 0$;

- type (2,1,1) if $\Psi_0 = \Psi_1 = 0, \Psi_2 \neq 0$;

- type (3,1) if $\Psi_0 = \Psi_1 = \Psi_2 = 0, \Psi_3 \neq 0$;

- type (4) if $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \Psi_4 \neq 0$;

- type (2,2) if $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2 \neq 0$;

-type (-) or flat if all coefficients are zero.

### 1.4.1 Comment on type (3,1) space-times

For later use, it is interesting to note the following relation between vectors $U$ and $\Theta$, defined above by relations 1.31, 1.32 and 1.33, and algebraically special type (3,1) spaces. We claim that $\Theta_{[a;b]} + U_{[a}\Theta_{b]} = 0$ if, and only if, the space-time is of type (3,1) or more degenerate.

To prove this claim we first calculate the second derivative of the bivector $N$ defined by 1.22 and find

$$\begin{aligned} N_{ab;cd} &= N_{ab}U_{c;d} + M_{ab}\Theta_{c;d} \\ &\quad + \left(N_{ab}U_d + M_{ab}\Theta_d\right)U_c \\ &\quad +2\left(N_{ab}X_d + L_{ab}\Theta_d\right)\Theta_c \end{aligned} \tag{1.65}$$

where we used relations 1.31 and 1.32 for derivatives of $N$ and $M$ respectively. We can rewrite this last expression as

$$\begin{aligned} N_{ab;cd} &= N_{ab}\left(U_{c;d} + U_c U_d + 2\Theta_c X_d\right) \\ &\quad + M_{ab}\left(\Theta_{c;d} + U_c \Theta_d\right) \\ &\quad + 2L_{ab}\Theta_c\Theta_d \end{aligned} \tag{1.66}$$

and remark that taking the antisymmetric part on $(c, d)$ transforms it into

$$N_{ab;[cd]} = N_{ab}\left(U_{[c;d]} + 2\Theta_{[c}X_{d]}\right) + M_{ab}\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right). \tag{1.67}$$

Note that the expression we are interested in is exactly the coefficient of $M_{ab}$ in the last expression. The left hand-side can be written

$$N_{ab;[cd]} = \frac{1}{2}\left(N_{ab;cd} - N_{ab;dc}\right) = \frac{1}{2}\left(N_{ap}R^p{}_{bcd} - N_{bp}R^p{}_{acd}\right) \tag{1.68}$$

where $R$ is the curvature tensor and we used the fact that $N$ is antisymmetric in $(a, b)$. We then have

$$N_{ap}R^p{}_{qcd} - N_{qp}R^p{}_{acd} = 2N_{aq}\left(U_{[c;d]} + 2\Theta_{[c}X_{d]}\right) + 2M_{aq}\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right) \tag{1.69}$$

where, for mere convenience, we replaced the index $b$ by $q$. Contracting this last expression by $N_b{}^q$ and using the fact that $N_{aq}N^{qb} = 0$ and $N_{aq}M^q{}_b = -N_{ab}$ we find

$$N_a{}^p N_b{}^q R_{pqcd} = 2N_{ab}\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right). \tag{1.70}$$

Remark that the $(c, d)$ pair of indices is just decorative, it doesn't enter into calculations at all. In fact, we can denote $R_{pqcd}$ by $\widehat{R}_{pq}$ and view the left hand-side as an operation between three 2-index objects. Also remark that the self-dual bivector $N$ picks out the self-dual part of $\widehat{R}_{pq}$ hence this last tensor can be regarded as a self-dual one.

Next we use one important identity from basic algebra of bivectors (see for example [4]): for any two bivectors $F$ and $G$ we have

$$F_{aq}G^q{}_b - G^*_{aq}F^{*q}{}_b = \left(F_{pq}G^{qp}\right)\delta_{ab}. \tag{1.71}$$

However, we are dealing with self-dual bivectors which have the additional property that $X^*_{ab} = -iX_{ab}$ hence the previous relation becomes

$$F_{aq}G^q{}_b + G_{aq}F^q{}_b = \left(F_{pq}G^{qp}\right)\delta_{ab}. \tag{1.72}$$

Using this for the left hand-side of relation 1.70 we find

$$\begin{aligned} N_a{}^p N_b{}^q \widehat{R}_{pq} &= N_a{}^p \left[\widehat{R}_{bq}N^q{}_p + \left(N_{st}\widehat{R}^{st}\right)\delta_{bp}\right] \\ &= \left(N_{st}\widehat{R}^{st}\right) N_{ab} \end{aligned} \tag{1.73}$$

hence 1.70 becomes

$$N_{ab}\left(N_{st}R^{st}{}_{cd}\right) = 2N_{ab}\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right) \tag{1.74}$$

or

$$N_{st}R^{st}{}_{cd} = 2\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right). \tag{1.75}$$

Taking into consideration that we work in an empty space, we can write the last relation with the Weyl tensor instead of the curvature tensor

$$N_{st}C^{st}{}_{cd} = 2\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right). \tag{1.76}$$

Again we use the fact that the self-dual bivector $N$ picks out the (left) self-dual part of the Weyl tensor so we can rewrite the last expression as

$$N_{st}{}^+C^{st}{}_{cd} = 2\left(\Theta_{[c;d]} + U_{[c}\Theta_{d]}\right). \tag{1.77}$$

The claim is now obvious since $N_{st}{}^+C^{st}{}_{cd} = 0$ if, and only if, $N$ is a PNB of multiplicity at least three (see also relations 1.60 and 1.63 as well as the Petrov classification given afterwards).

### 1.5 Robinson-Trautman solutions

Robinson-Trautman solutions are algebraically special space-times constructed around a geodesic, shear-free, non-twisting but expanding null congruence. In complex coordinates $\rho$, $\sigma$, $\zeta$, $\overline{\zeta}$, the line element for this class of space-times can be written as (see [22])

$$ds^2 = 2d\rho d\sigma + cd\sigma^2 - \frac{2\rho^2}{p^2}d\zeta d\overline{\zeta} \tag{1.78}$$

where

$$p = p(\sigma, \zeta, \overline{\zeta}), \tag{1.79}$$

$$c = -\frac{2m}{\rho} + K - 2H\rho, \tag{1.80}$$

$$m = m(\sigma), \tag{1.81}$$

$$H = p^{-1}p_\sigma, \tag{1.82}$$

$$K = 2(pp_{\zeta\overline{\zeta}} - p_\zeta p_{\overline{\zeta}}). \tag{1.83}$$

There is only one field equation to be satisfied

$$\Delta K = 4(m_\sigma - 3Hm), \tag{1.84}$$

where $\Delta = 2p^2\partial_\zeta\partial_{\overline{\zeta}}$.

One can choose a null tetrad to be

$$\begin{aligned} n_a &= \sigma,_a \\ l_a &= \rho,_a + \frac{1}{2}c\sigma,_a \\ m_a &= \frac{\rho}{p}\zeta,_a . \end{aligned} \tag{1.85}$$

The metric can then be written as

$$g_{ab} = n_a l_b + l_a n_b - m_a\overline{m}_b - \overline{m}_a m_b. \tag{1.86}$$

In terms of the bivectors

$$N_{ab} = n_a m_b - m_a n_b, \tag{1.87}$$

$$M_{ab} = n_a l_b - l_a n_b + m_a \overline{m}_b - \overline{m}_a m_b, \tag{1.88}$$

$$L_{ab} = \overline{m}_a l_b - l_a \overline{m}_b, \tag{1.89}$$

the self dual Weyl tensor is

$$\begin{aligned} 2{}^{+}C_{abcd} = & \frac{2m}{\rho^2}\left(N_{ab}L_{cd} + L_{ab}N_{cd} + M_{ab}M_{cd}\right) \\ & + \frac{p}{\rho^2}K_\zeta\left(N_{ab}M_{cd} + M_{ab}N_{cd}\right) - \frac{1}{\rho^2}\left(p^2 c_\zeta\right)_{,\zeta} N_{ab}N_{cd}. \end{aligned} \tag{1.90}$$

Comparing 1.54 and 1.90 we see that for Robinson-Trautman spaces the coefficients of the self-dual Weyl tensor are

$$\Psi_2 = \frac{m}{\rho^2} \tag{1.91}$$

$$\Psi_3 = \frac{1}{2}\frac{p}{\rho^2}K_\zeta \tag{1.92}$$

$$\Psi_4 = -\frac{1}{2}\frac{1}{\rho^2}\left(p^2 c_\zeta\right)_{,\zeta}. \tag{1.93}$$

We are going to use these expressions later to calculate some invariants for Robinson-Trautman spaces. Remark that these spaces are of type III or more degenerate if, and only if, $m = 0$.

## 1.6 Solutions with twisting rays

A space-time admits a geodesic, shear-free twisting and diverging null congruence if the line element can be written in $u$, $r$, $\zeta$, $\overline{\zeta}$ coordinates as (see [6], [3], [24], [36])

$$ds^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4 \tag{1.94}$$

with

$$\omega^1 = \frac{r - i\Sigma}{p^2} d\zeta = \overline{\omega}^2 \tag{1.95}$$

$$\omega^3 = du + Ld\zeta + \overline{L}d\overline{\zeta} \tag{1.96}$$

$$\omega^4 = dr + Wd\zeta + \overline{W}d\overline{\zeta} + H\omega^3 \tag{1.97}$$

where the metric functions satisfy the conditions

$$2i\Sigma = p^2 \left(\overline{\partial}L - \partial\overline{L}\right), \tag{1.98}$$

$$W = -L_{,u}(r + i\Sigma) + i\partial\Sigma, \tag{1.99}$$

$$\partial = \partial_\zeta - L\partial_u, \tag{1.100}$$

$$H = -r(\ln p)_{,u} - \frac{mr + M\Sigma}{r^2 + \Sigma^2} + \frac{K}{2}, \tag{1.101}$$

$$K = 2p^2 Re\left[\partial\left(\overline{\partial}\ln p - \overline{L}_{,u}\right)\right], \tag{1.102}$$

$$M = \Sigma K + p^2 Re\left[\partial\overline{\partial}\Sigma - 2\overline{L}_{,u}\partial\Sigma - \Sigma\partial_u\partial\overline{L}\right] \tag{1.103}$$

and the remaining field equations

$$\partial(m + iM) = 3(m + iM)L_{,u} \tag{1.104}$$

$$\left[p^{-3}(m + iM)\right]_{,u} = p\left[\partial + 2(\partial\ln p - L_{,u})\right]\partial I \tag{1.105}$$

where

$$I = \overline{\partial}\left(\overline{\partial}\ln p - \overline{L}_{,u}\right) + \left(\overline{\partial}\ln p - \overline{L}_{,u}\right)^2 \tag{1.106}$$

Remark that all the metric function depend only on the two main functions $p$ and $L$; this means that finding solutions of this sort is equivalent with finding functions $p$ and $L$ satisfying the field equations.

By the introduction of a *potential* $V$ for the function $p$ (see [23])

$$V_{,u} = p \tag{1.107}$$

the field equations 1.103, 1.104 and 1.105 formally simplify to

$$\left[p^{-3}\left(m+iM\right)-\partial\partial\overline{\partial\partial}V\right]{,_u} = -p^{-1}\left(\partial L\right){,_u}\left(\overline{\partial L}\right){,_u} \tag{1.108}$$

$$\partial\left(m+iM\right) = 3\left(m+iM\right)L{,_u} \tag{1.109}$$

$$p^{-3}M = Im\overline{\partial\partial}\partial L \tag{1.110}$$

respectively. Kerr introduced in [6] $p=1$ gauge in which the field above equations take the form

$$\left[\left(m+iM\right)+\partial\partial\overline{\partial L}\right]{,_u} = -\left(\partial L\right){,_u}\left(\overline{\partial L}\right){,_u} \tag{1.111}$$

$$\partial\left(m+iM\right) = 3\left(m+iM\right)L{,_u} \tag{1.112}$$

$$M = Im\overline{\partial\partial}\partial L. \tag{1.113}$$

We are going to use $p=1$ gauge and the form of the field equations in this gauge in Chapter 5.

As an interesting fact, we remark that the Robinson-Trautman solutions, described in the previous section, are a special case of the twisting ones: they are obtained by putting $L=0$ in the metric functions and the field equations written above.

In terms of the function $I$, defined in 1.106, the remaining components of the Weyl tensor take the form given by Trim and Wainwright in [36]

$$\Psi_2 = \psi_2^0\rho^3 \tag{1.114}$$

$$\Psi_3 = \psi_3^0\rho^2+Y_1\rho^3+Y_2\rho^4 \tag{1.115}$$

$$\Psi_4 = \psi_4^0\rho+Z_1\rho^2+\frac{1}{2}Z_2\rho^2+\frac{1}{3}Z_3\rho^3+\frac{1}{4}Z_4\rho^4 \tag{1.116}$$

where $\rho = -\frac{1}{r+i\Sigma}$ is the complex expansion and

$$\psi_2^0 = m + iM \tag{1.117}$$

$$\psi_3^0 = -p^3 \partial I \tag{1.118}$$

$$Y_1 = 2p\left(\partial - 3L_{,u}\right)\psi_2^0 \tag{1.119}$$

$$Y_2 = -3N\psi_2^0 \tag{1.120}$$

$$\psi_4^0 = p^2 \partial_u I \tag{1.121}$$

$$Z_1 = 2\left(\partial - 4L_{,u}\right)\left(p\psi_3^0\right) \tag{1.122}$$

$$Z_2 = 2\left(\partial - 4L_{,u}\right)\left(pY_1\right) + 4N\psi_3^0 \tag{1.123}$$

$$Z_3 = 2\left(\partial - 5L_{,u}\right)\left(pY_2\right) + 6NY_1 \tag{1.124}$$

$$Z_4 = 8NY_2 \tag{1.125}$$

$$N = 2ip\left(\partial - 3L_{,u}\right)\Sigma. \tag{1.126}$$

The degree of algebraic specialty of these solutions depends on the degree of nullity of these components (which can be seen from 1.4). For example a solution of type (3,1) is one that has $m + iM = 0$ and $\partial I \neq 0$.

Also remark that $\psi_2^0 = 0$ (algebraic type (3,1) or more degenerate) implies that $Y_1 = Y_2 = Z_3 = Z_4 = 0$ and if $\psi_2^0 = \psi_3^0 = 0$ (algebraic type (4) or flat) the only nonzero coefficient is $\psi_4^0$.

# CHAPTER 2

# CLASSIFICATION OF GRAVITATIONAL FIELDS BY MEANS OF INVARIANTS

This chapter describes a classification of algebraically special gravitational fields by means of algebraic and differential invariants. The only two algebraic invariants of the Weyl tensor are given in Section 1.1; they achieve only a partial characterization of algebraically special Weyl tensor. To obtain a complete one, one has to turn to differential invariants. This is done systematically in Section 1.2.

It is worth remarking that we are considering only expanding space-times ($\rho \neq 0$ where $\rho$ is the complex expansion). This approach makes sense since it has been shown in [1] and [19] that the differential invariants of any order vanish for type III and type N space-times with zero expansion.

The importance of these invariants exceeds the classification purpose: they also provide a measure of the amplitude of the gravitational field. And for the (2,1,1) case, the null tensor $A$ defined below provides a measure of the deviation of the solution from a (2,2) one.

## 2.1 Algebraic Invariants

There are only two algebraic invariants that one can form from the self dual Weyl tensor, namely

$$I = \frac{1}{4} {}^{+}C_{abcd} {}^{+}C^{abcd}, \tag{2.1}$$

$$J = \frac{1}{8} {}^{+}C_{abcd} {}^{+}C^{cd}{}_{ef} {}^{+}C^{efab}. \tag{2.2}$$

Higher order combinations of ${}^{+}C$ will linearly depend on $I$ and $J$. This is best seen in terms of the 3x3 symmetric complex matrix $Q$ associated with the Weyl tensor as follows.

It has long been known (see for example Synge's "Special relativity" [32]) that for any given vector $t_a$ and any bivector $F_{ab}$ one can define

$$E_a = F_{ab} t^b \tag{2.3}$$

$$B_a = F^{*}_{ab} t^b \tag{2.4}$$

with the obvious conclusions that $E_a t^a = 0$ and $B_a t^a = 0$ and the not so obvious one that

$$t_r t^r F_{ab} = 2E_{[a} t_{b]} - 2^{*} \left( B_{[a} t_{b]} \right). \tag{2.5}$$

The converse is also true: given $E$, $B$ and $t$ satisfying $E_a t^a = 0$ and $B_a t^a = 0$ one can define the bivector $F$ by equation 2.5. Hence, modulo the vector $t$, one has a 1 to 2 correspondence between the space of bivectors and the 3-dimensional space of vectors orthogonal to $t$.

Next, consider $t$ to be a unit time-like vector and suppose $F$ is self-dual i.e. $F = {}^{+}F$ (note that $F$ has to be complex in this case). But this implies that $F^{*} = -iF$ hence

$$E_a = iB_a \tag{2.6}$$

which implies further that the 1 to 2 correspondence mentioned earlier becomes a one-to-one correspondence. So, modulo $t$, we have achieved an isomorphism between self-dual bivector space $({}^{+}\Lambda^2)$ and 3-dimensional complex vector space $(V)$. This implies that the spaces of homomorphisms of the two are also isomorphic

$$\hom({}^{+}\Lambda^2) \sim \hom(V)\,. \tag{2.7}$$

Remark that an element of $\hom({}^{+}\Lambda^2)$ is a four-index tensor which is antisymmetric and self-dual on the pair of indices $(ab)$ and $(cd)$; the explicit isomorphism can be written in terms of

$$E_{ac} = F_{abcd}u^b u^d \tag{2.8}$$

as

$$-F_{abcd} = 4u_{[a}E_{b][d}u_{c]} + g_{a[c}E_{d]b} - g_{b[c}E_{d]a} + i\varepsilon_{abef}u^e u_{[c}E_{d]}{}^f + i\varepsilon_{cdef}u^e u_{[a}E_{b]}{}^f. \tag{2.9}$$

(more details about this isomorphism can be found in [2]). Let $Q$ be the matrix associated with the self-dual part of the Weyl tensor via this isomorphism. It is easy to see that the matrix $Q$ is symmetric $(Q_{ab} = Q_{ba})$ and trace-free $(Q_a{}^a = 0)$ hence it has 5 complex independent components.

We have

$$Q^2 = \frac{1}{2}{}^{+}C_{abcd}{}^{+}C^{cd}{}_{ef} \quad , \tag{2.10}$$

$$Q^3 = \frac{1}{4}{}^{+}C_{abcd}{}^{+}C^{cd}{}_{ef}{}^{+}C^{ef}{}_{gh} \tag{2.11}$$

and so on, which implies

$$traceQ = {}^{+}C_{ab}{}^{ab} = 0 \tag{2.12}$$

$$traceQ^2 = \frac{1}{4}{}^{+}C_{abcd}{}^{+}C^{abcd} = I \tag{2.13}$$

$$traceQ^3 = \frac{1}{8}{}^{+}C_{abcd}{}^{+}C^{cd}{}_{ef}{}^{+}C^{abef}. \tag{2.14}$$

The characteristic polynomial of $Q$ is (see [16])

$$p = 6x^3 - 3Ix - 2J \tag{2.15}$$

and from Hamilton-Cayley theorem we have $p(Q) = 0$ which doesn't allow independent higher powers of $Q$.

Next, let $x_1$, $x_2$ and $x_3$ be the eigenvalues of the matrix $Q$. Relations 2.12, 2.13 and 2.14 translate into

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1^2 + x_2^2 + x_3^2 &= I \\ x_1^3 + x_2^3 + x_3^3 &= J. \end{aligned} \tag{2.16}$$

Cubing the first relation and subtracting three times the product of the first two we find

$$J = 3x_1x_2x_3 \tag{2.17}$$

which means the scalar $J$ is proportional to the determinant of the matrix $Q$; using this last relation along with the cube of the second one in 2.16 we find

$$I^3 - 6J^2 = 2\left(x_1 - x_2\right)^2\left(x_2 - x_3\right)^2\left(x_3 - x_1\right)^2 \tag{2.18}$$

which implies that two or more of the eigenvalues of the matrix $Q$ coincide if and only if $I^3 = 6J^2$. The coincidence of these eigenvalues is in close relation with the algebraic types of the Weyl tensor (more details can be found in [16]): we can distinct three classes of solutions

- $I^3 \neq 6J^2$ containing solutions of type (1,1,1,1);

- $I^3 = 6J^2 \neq 0$ containing solutions of type (2,1,1) and (2,2);

- $I = J = 0$ containing solutions of type (3,1), (4) and (-) or flat.

This is the best classification one can get using algebraic invariants alone. To distinct between different types that fall under the same class, one has to use differential invariants.

## 2.2 Differential Invariants

To complete the classification we need one differential invariant to distinct type (3,1) from more degenerate ones, another one to distinct type (4) from flat space-times and yet another one to distinct type (2,1,1) from (2,2). This will be done in the following subsections. It is interesting to remark that these distinctions can all be expressed in terms of null, or type (4), tensors and therefore they can be expressed in terms of type (4) invariant

$$J_F = F_{abcd;rs} F^{abcd}{}_{;tu} \overline{F}_{efgh;}{}^{rs} \overline{F}^{efgh;tu} \tag{2.19}$$

where $F$ is a null 4-tensor. Remark that $J_F$ is the invariant $J$ in [14]. We are particularly interested in $J_A$, $J_B$ and $J_{+C}$ where

$$A_{abcd} = I B_{abcd} - J {}^{+}C_{abcd} \tag{2.20}$$

$$B^{ab}{}_{cd} = \frac{1}{2} {}^{+}C^{ab}{}_{rs} {}^{+}C^{rs}{}_{cd} - \frac{1}{3} I {}^{+}\delta^{ab}_{cd} \tag{2.21}$$

where ${}^{+}\delta_{abcd} = \frac{1}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd} - i\eta_{abcd}\right)$, $\eta$ being the Levi-Civita tensor.

Although all the quantities defined above are independent of any coordinate system, it is easier to understand their role if we look at them from the point of view of an adapted null tetrad as defined in Chapter 1.

### 2.2.1 (3,1) case

A type III, self-dual Weyl tensor can be written as (see 1.54)

$$^{+}C_{abcd} = \Psi_3\left(N_{ab}M_{cd} + M_{ab}N_{cd}\right) + \Psi_4 N_{ab}N_{cd}. \tag{2.22}$$

Making use of the products $N_{ab}N^{ab} = 0$, $M_{ab}M^{ab} = -4$ and the fact that $I = 0$ we get

$$B_{abcd} = -\frac{1}{2}\left(2\Psi_3\right)^2 N_{ab}N_{cd}. \tag{2.23}$$

To calculate $J_B$ we first remark that

$$B_{abcd;rs}B^{abcd}{}_{;tu} = \frac{1}{2}\left(2\Psi_3\right)^4\left(N_{ab;r}N^{ab}{}_{;t}N_{cd;s}N^{cd}{}_{;u} + N_{ab;r}N^{ab}{}_{;u}N_{cd;s}N^{cd}{}_{;t}\right) \tag{2.24}$$

Then, since $N_{ab;r}N^{ab}{}_{;t} = -4\Theta_r\Theta_t$, we find

$$B_{abcd;rs}B^{abcd}{}_{;tu} = \left(4\Psi_3\right)^4\Theta_r\Theta_s\Theta_t\Theta_u \tag{2.25}$$

hence

$$J_B = \left|4\Psi_3\right|^8\left(\Theta_r\overline{\Theta}^r\right)^4. \tag{2.26}$$

However, $\Theta_r\overline{\Theta}^r = -\rho\overline{\rho}$,where $\rho$ is the complex expansion of the principal null direction, so the final expression for $J_B$ is

$$J_B = \left|4\Psi_3\rho\right|^8. \tag{2.27}$$

Remark that $J_B \neq 0 \Leftrightarrow \Psi_3 \neq 0 \Longleftrightarrow (3,1)$. If $J_B = 0$ the Weyl tensor degenerates into a (4) or a (-) type.

### 2.2.2 (4) case

A type N, self dual Weyl tensor is

$$^{+}C_{abcd} = \Psi_4 N_{ab} N_{cd}. \tag{2.28}$$

Remark that, as before, we are dealing with a null tensor. Consequently we find

$$^{+}C_{abcd;rs}\,^{+}C^{abcd}{}_{;tu} = 2\Psi_4^2 \left( N_{ab;r} N^{ab}{}_{;t} N_{cd;s} N^{cd}{}_{;u} + N_{ab;r} N^{ab}{}_{;u} N_{cd;s} N^{cd}{}_{;t} \right) \tag{2.29}$$

hence

$$^{+}C_{abcd;rs}\,^{+}C^{abcd}{}_{;tu} = (8\Psi_4)^2\, \Theta_r \Theta_s \Theta_t \Theta_u \tag{2.30}$$

which implies

$$J_{^{+}C} = |8\Psi_4|^4 \left( \Theta_r \overline{\Theta}^r \right)^4 \tag{2.31}$$

or

$$J_{^{+}C} = \left| 8\Psi_4 \rho^2 \right|^4 \tag{2.32}$$

where $\rho$, as before, is the complex expansion of the principal null direction. Obviously, $J_{^{+}C} \neq 0 \Longleftrightarrow \Psi_4 \neq 0 \Longleftrightarrow$ at least type (4). If $J_{^{+}C} = 0$ we have flat space-time.

### 2.2.3 (2,1,1) case

A type (2,1,1) Weyl tensor can be written as

$$\begin{aligned} ^{+}C_{abcd} = {} & \Psi_2 \left( N_{ab} L_{cd} + L_{ab} N_{cd} + M_{ab} M_{cd} \right) + \\ & \Psi_3 \left( N_{ab} M_{cd} + M_{ab} N_{cd} \right) + \Psi_4 N_{ab} N_{cd}. \end{aligned} \tag{2.33}$$

Again, making use of the various products between bivectors, one calculates

$$
\begin{aligned}
{}^{+}C_{abrs}{}^{+}C^{rs}{}_{cd} &= 2\Psi_2^2\left(N_{ab}L_{cd}+L_{ab}N_{cd}-2M_{ab}M_{cd}\right) \\
&\quad -2\Psi_2\Psi_3\left(N_{ab}M_{cd}+M_{ab}N_{cd}\right) \\
&\quad +4\left(\Psi_2\Psi_4-\Psi_3^2\right)N_{ab}N_{cd}
\end{aligned}
\tag{2.34}
$$

which, along with

$$
{}^{+}\delta^{ab}_{cd} = N_{ab}L_{cd}+L_{ab}N_{cd}-\frac{1}{2}M_{ab}M_{cd}, \tag{2.35}
$$

and

$$
I = 6\Psi_2^2 \tag{2.36}
$$

implies

$$
\begin{aligned}
B_{abcd} &= -\Psi_2^2\left(N_{ab}L_{cd}+L_{ab}N_{cd}+M_{ab}M_{cd}\right) \\
&\quad -\Psi_2\Psi_3\left(N_{ab}M_{cd}+M_{ab}N_{cd}\right) \\
&\quad +2\left(\Psi_2\Psi_4-\Psi_3^2\right)N_{ab}N_{cd}.
\end{aligned}
\tag{2.37}
$$

This last expression and the fact that $J=-6\Psi_2^3$ implies that

$$
A_{abcd} = 6\Psi_2^2(3\Psi_2\Psi_4-2\Psi_3^2)N_{ab}N_{cd}. \tag{2.38}
$$

This null tensor $A$ can be viewed as a measures of deviation of a (2,1,1) solution from a (2,2) one. It is easy to see now that

$$
J_A = \left|6\Psi_2^2(3\Psi_2\Psi_4-2\Psi_3^2)\rho^2\right|^4. \tag{2.39}
$$

We can simplify this expression by performing a complex transformation on $N$, $M$ and $L$ bivectors which preserves all the algebraic properties of the space-time. The three bivectors are fixed up to

$$\begin{aligned} L &\longrightarrow L + \mu M + \mu^2 N \\ M &\longrightarrow M + 2\mu N \\ N &\longrightarrow N \end{aligned} \tag{2.40}$$

where $\mu$ is an arbitrary complex parameter; and if we choose $\mu = -\frac{\Psi_3}{3\Psi_2}$ the coefficient of $NM + MN$ in the expression of ${}^+C_{abcd}$ vanishes. So without any loss of generality we can take $\Psi_3 = 0$ in 2.33, 2.38 and 2.39 above. The simplified expression of $J_A$ is

$$J_A = \left|18\Psi_2^3\Psi_4\rho^2\right|^4; \tag{2.41}$$

it is then easily seen that $J_A \neq 0 \Longleftrightarrow \Psi_4 \neq 0 \Longleftrightarrow$ type (2,1,1). If $J_A = 0$ we have a (2,2) space-time.

## 2.3 Conclusion

For space-times admitting an expanding congruence we have achieved the following classification:

- $I^3 \neq 6J^2,\ I \neq 0,\ J \neq 0$ : (1,1,1,1);
- $I^3 = 6J^2 \neq 0,\ J_A \neq 0$ : (2,1,1);
- $I^3 = 6J^2 \neq 0,\ J_A = 0$ : (2,2);
- $I = J = 0,\ J_B \neq 0$ : (3,1);
- $I = J = 0,\ J_B = 0,\ J_{+C} \neq 0$ : (4);
- $I = J = 0,\ J_B = 0,\ J_{+C} = 0$ : (-).

# CHAPTER 3

# TYPE III SPACE-TIMES: INVARIANTS AND REGULARITY

The local properties of the gravitational field can be described by the curvature tensor and its covariant derivatives to different orders. These properties will show up in scalars formed from them by contracting their tensor products. In particular, the appearance of singularities in such scalars is an indication of a local singularity in the field. The converse is not true: the mere absence of singularities in these scalars is no proof of regularity of the field. For example the C-metric describes a space-time which is singular although all the known invariants are regular.

This chapter focuses on type III space-times from the point of view of invariants. One invariant was already given in Chapter 2 where it was used to classify algebraically special gravitational fields. However, this invariant is neither the simplest nor the only one that it exists. In Section 2 we show the construction of an invariant of order one (i.e. using only first order derivatives of the Weyl tensor) and its expressions for the known solutions available. It will immediately be apparent that they contain singularities. Section 3 discusses another invariant of order 2, some comments being made about the expression of it for a Robinson-Trautman type III solution.

## 3.1 An invariant of the first order

Let $n_a$, $l_a$, $m_a$, $\overline{m}_a$ be a null tetrad, as described in Chapter 1, and $N$, $M$, $L$ the self-dual bivectors defined from it. The self-dual part of a type III Weyl tensor has then the following expression

$$^{+}C_{abcd} = \Psi_3 \left(N_{ab}M_{cd} + M_{ab}N_{cd}\right) + \Psi_4 N_{ab}N_{cd}. \tag{3.1}$$

All scalars formed using undifferentiated $^{+}C$ alone will vanish. To obtain a nonzero quantity we have to use the first covariant derivative

$$\begin{aligned} ^{+}C_{abcd;s} = {} & N_{ab}N_{cd}\left(\Psi_{4,s} + 4\Psi_3 X_s + 2\Psi_4 U_s\right) \\ & \left(N_{ab}M_{cd} + M_{ab}N_{cd}\right)\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4 \Theta_s\right) \\ & + 2\Psi_3 M_{ab}M_{cd}\Theta_s \\ & + 2\Psi_3\left(N_{ab}L_{cd} + L_{ab}N_{cd}\right)\Theta_s \end{aligned} \tag{3.2}$$

where we used relations 1.31 and 1.32 for first order derivatives of the fundamental bivectors $N$ and $M$.

Using the fact that $N_{ab}L^{ab} = 2$, $M_{ab}M^{ab} = -4$ and that all other products of this sort are zero, we find that the contraction of this first derivative and the undifferentiated $^{+}C$ will vanish. Next, try contracting $^{+}C_{abcd;s}$ by $^{+}C^{abcd}{}_{;t}$; because of the $M_{ab}M_{cd}$ and $N_{ab}L_{cd} + L_{ab}N_{cd}$ terms this will be a nonzero quantity. We find

$$^{+}C_{abcd;s}{}^{+}C^{abcd}{}_{;t} = 6\left(4\Psi_3\right)^2 \Theta_s\Theta_t. \tag{3.3}$$

Contracting next by its complex conjugate we find

$$J_1 = 6^2 \left|4\Psi_3\right|^4 \left(\Theta_s\overline{\Theta}^s\right)^2. \tag{3.4}$$

Next recall that

$$\Theta_c = -\tau n_c - \kappa l_c + \rho m_c + \sigma \overline{m}_c \tag{3.5}$$

where $\tau$, $\kappa$, $\rho$, $\sigma$ are four of the twelve complex Newman-Penrose spin coefficients. Then, in general,

$$\Theta_s\overline{\Theta}^s = \tau\overline{\kappa} + \overline{\tau}\kappa - (\rho\overline{\rho} + \sigma\overline{\sigma})\,. \tag{3.6}$$

However, $n$ is geodesic and shear-free, which implies that $\kappa = 0$ and $\sigma = 0$ which transforms the last relation into

$$\Theta_s\overline{\Theta}^s = -\rho\overline{\rho} = -\left|\rho\right|^2. \tag{3.7}$$

The invariant $J_1$ takes then the form

$$J_1 = 6^2\left|4\rho\Psi_3\right|^4. \tag{3.8}$$

Remark that, for expanding space-times, this invariant might as well be used instead of $J_A$ to distinct algebraic type (3,1) of the Weyl tensor from more degenerate ones.

### 3.1.1 Robinson-Trautman type III spaces

Robinson-Trautman spaces were described in Chapter 1 section 1.5 as being those constructed around a geodesic and shear-free, non-twisting but expanding null congruence. From the expression 1.90 of the self dual Weyl tensor, one can see that a necessary and sufficient condition for these spaces to be of type III (or more degenerate) is $m = 0$. All the other quantities involved in the definition of the space-time depend then on the function $p$ alone.

The only known type III spaces of this sort are given by $p = \left(F + \overline{F}\right)^{3/2}\left|F_\zeta\right|^{-1}$ where $F = F(\zeta, u)$ is an arbitrary analytic function. Recall that

$$\Psi_3 = \frac{1}{2}\frac{p}{\rho^2}K_\zeta \tag{3.9}$$

where $K = 2(pp_{\zeta\overline{\zeta}} - p_\zeta p_{\overline{\zeta}})$. Plugging in the expression for $p$ one finds

$$K = -3(F + \overline{F}) \tag{3.10}$$

hence

$$\Psi_3 = -\frac{3}{2}\rho^2 \left(F + \overline{F}\right)^{3/2} \left(\frac{F_\zeta}{\overline{F}_{\overline{\zeta}}}\right)^{1/2}. \tag{3.11}$$

The calculation of the invariant $J_1$ is now easy. We find

$$J_1 = \left[6\left(F + \overline{F}\right)|\rho|^2\right]^6. \tag{3.12}$$

Using Liouville's theorem one sees that the function $F$ cannot be bounded, hence, at least at some point, the invariant will blow up.

### 3.1.2 Twisting type III spaces

The only known type III space-times built around a twisting, geodesic and shear free null congruence are those obtained from "twisted up" Robinson Trautman solutions as described in [25] or [8]. Since there is only one class of such Robinson-Trautman solution we have one corresponding twisting solution given by $p = (\zeta + \overline{\zeta})^{3/2}$, $L_{,u} = 0$. In this case we have (see [8])

$$\Psi_3 = -p^3\rho^2\partial I \tag{3.13}$$

where $I = \overline{\partial\partial}\ln p + \left(\overline{\partial}\ln p\right)^2$. Since $p$ doesn't depend on $u$, the expression for $I$ simplifies to become $I = \frac{p_{\overline{\zeta}\zeta}}{p}$ which implies

$$\Psi_3 = -\rho^2 p\left(pp_{\overline{\zeta\zeta}\zeta} - p_{\overline{\zeta\zeta}}p_\zeta\right). \tag{3.14}$$

Substituting $p = (\zeta + \overline{\zeta})^{3/2}$ in this last expression we find

$$\Psi_3 = \frac{3}{2}\rho^2(\zeta + \overline{\zeta})^{3/2} \tag{3.15}$$

hence the invariant $J_1$ is

$$J_1 = \left[6\left(\zeta + \overline{\zeta}\right)^6 |\rho|^2\right]^6 . \tag{3.16}$$

The resemblance between formulae 3.12 and 3.16 is not random. It appears as a consequence of the way the twisted solution is derived from the Robinson-Trautman one. As before, we can make the immediate remark that $J_1$ is singular.

## 3.2 An invariant of the second order

This chapter doesn't investigate systematically invariants for type III space-times. Besides the invariant of order one presented in the previous section,we suspect there is a large number of invariants of order two already. Invariants of higher order become too complicated to be useful. However, in this section we present another invariant of order two and calculate its expression for the simple case of a Robinson-Trautman solution. Why bother with this new one when we have already shown the singularity of these solutions? For the simple reason that it gives a different type of information about them.

We work in the same null tetrad in which the Weyl tensor is given by

$$^{+}C_{abcd} = \Psi_3\left(N_{ab}M_{cd} + M_{ab}N_{cd}\right) + \Psi_4 N_{ab}N_{cd}. \tag{3.17}$$

Using, as before, relations 1.31, 1.32 and 1.33 we find

$$\begin{aligned} ^{+}C_{abcd;s} = {} & N_{ab}N_{cd}\left(\Psi_{4,s} + 4\Psi_3 X_s + 2\Psi_4 U_s\right) \\ & + \left(N_{ab}M_{cd} + M_{ab}N_{cd}\right)\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4 \Theta_s\right) \\ & + 2\Psi_3 M_{ab}M_{cd}\Theta_s \\ & + 2\Psi_3\left(N_{ab}L_{cd} + L_{ab}N_{cd}\right)\Theta_s \end{aligned} \tag{3.18}$$

and

$$
\begin{aligned}
{}^{+}C_{abcd;st} \quad = \quad & N_{ab}N_{cd}[\Psi_{4},_{st} + 4\left(\Psi_3 X_s\right),_t + 2\left(\Psi_4 U_s\right),_t + 2U_t\left(\Psi_{4,s} + 4\Psi_3 X_s + 2\Psi_4 U_s\right) \\
& + 4X_t\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4 \Theta_s\right)] \\
& + \left(N_{ab}M_{cd} + M_{ab}N_{cd}\right)\left[\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4 \Theta_s\right)_{;t} + \Theta_t\left(\Psi_{4,s} + 4\Psi_3 X_s + 2\Psi_4 U_s\right)\right. \\
& + U_t\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4 \Theta_s\right) + 3X_t\left(2\Psi_3\Theta_s\right)] \\
& + M_{ab}M_{cd}\left[\left(2\Psi_3\Theta_s\right)_{;t} + 2\Theta_t\left(\Psi_{3,s} + \Psi_3 U_s + \Psi_4\Theta_s\right)\right] \\
& + \left(M_{ab}L_{cd} + L_{ab}M_{cd}\right)3\Theta_t\left(2\Psi_3\Theta_s\right).
\end{aligned}
\tag{3.19}
$$

Let $D_{rst}$ be defined by

$$D_{rst} = {}^{+}C_{abcd;r}\,{}^{+}C^{abcd}{}_{;st}. \tag{3.20}$$

Substituting the expressions for ${}^{+}C_{abcd;s}$ and ${}^{+}C_{abcd;st}$ we find

$$
\begin{aligned}
D_{rst} = & -96\Psi_3\Theta_s\Theta_t\left(\Psi_{3},_r + \Psi_3 U_r + \Psi_4\Theta_r\right) \\
& + 96\Psi_3\Theta_r\left[\left(\Psi_3\Theta_s\right)_{;t} + \Theta_t\left(\Psi_{3},_s + \Psi_3 U_s + 2\Psi_4\Theta_s\right)\right].
\end{aligned}
\tag{3.21}
$$

Next calculate

$$
\begin{aligned}
D_{[rs]t} = & -192\Psi_3\left(\Psi_{3},_{[r}\Theta_{s]} + \Psi_3 U_{[r}\Theta_{s]}\right)\Theta_t \\
& + 96\Psi_3^2\Theta_{[r}\Theta_{s];t}.
\end{aligned}
\tag{3.22}
$$

and recall that for type (3,1) space-times we have $U_{[r}\Theta_{s]} = \Theta_{[s;r]}$ (see section 1.4.1) which eliminates the vector $U$ completely from the expression of $D_{[rs]t}$.

We obtain

$$D_{[rs]t} = -192\Psi_3\left(\Psi_3\Theta_{[s}\right)_{;r]}\Theta_t + 96\Psi_3^2\Theta_{[r}\Theta_{s];t}. \tag{3.23}$$

To get an invariant one has to contract the expression for $D_{[rs]t}$ by its complex conjugated

$$J_2 = D_{[rs]t}\overline{D}^{[rs]t}. \tag{3.24}$$

The general expression for $J_2$ is not a nice one. However $J_2$ can be computed for simple space-time; we carry on this calculation in the next section.

### 3.2.1 Robinson-Trautman type III spaces

The calculation of the invariant $J_2$ for Robinson-Trautman solutions was done with the help of GRTensor run under MAPLE V Release 4 on a Windows machine. The output is included at the end of this thesis as Appendix. The approach was to calculate the invariant in terms of a general $\Psi_3$ coefficient then to plug in the expression 1.92 for it and get the final result by hand calculations. All calculations were performed in real coordinates.

Robinson-Trautman in real coordinates is obtained via

$$\zeta = \frac{1}{\sqrt{2}}\left(\xi + i\eta\right), \tag{3.25}$$

so that the new coordinates are $r$, $\sigma$, $\xi$, $\eta$. We will start with the case $p = p(\xi)$, since the calculations are easier to follow ; the general one $p = p\left(\xi, \eta, \sigma\right)$ will follow.

*Case p=p(ξ)*

The invariant $J_2$ depends only on the real quantity $\Psi_3$ and it is found, using GRTensor (see Appendix), to be

$$J_2 = \frac{4608}{r^6}\Psi_3^2\left[\Psi_3^2 pp_{\xi\xi} + \left(\Psi_3 p_\xi\right)^2 + 2\left(p\Psi_{3,\xi}\right)^2 - 4pp_\xi\Psi_3\Psi_{3,\xi}\right]. \tag{3.26}$$

Substituting

$$\Psi_3 = \frac{1}{2\sqrt{2}}\frac{p}{r^2}K_\xi \tag{3.27}$$

and using the field equation $K_{\xi\xi}=0$ we find

$$J_2 = 72\frac{p^4}{r^{14}}KK_\xi^4. \tag{3.28}$$

Remark that this invariant can be written in terms of invariants obtained in [22] as

$$J_2 = 288\left(\frac{l}{r^4}\right)^2\frac{K}{r^2} \tag{3.29}$$

where $l=\frac{1}{2}\frac{p^2}{r^2}K_\xi^2$. This will no longer be true for the general $p=p(\xi,\eta,\sigma)$ case.

*Case p=p($\xi,\eta,\sigma$)*

In this case $\Psi_3$ is the complex function

$$\Psi_3 = \frac{1}{2\sqrt{2}}\frac{p}{r^2}\left(K_\xi - iK_\eta\right). \tag{3.30}$$

The expression for $J_2$, found in Appendix using GRTensor , is

$$\begin{aligned} J_2 = {} & \frac{4608}{r^6}\left|\Psi_3\right|^2\left[\left|\Psi_3\right|^2 p\left(p_{\xi\xi}+p_{\eta\eta}\right)+\left|\Psi_3\right|^2\left(p_\xi^2+p_\eta^2\right)+2p^2\left(\left|\Psi_{3,\xi}\right|^2+\left|\Psi_{3,\eta}\right|^2\right)\right. \\ & -2pp_\xi\left|\Psi_3\right|^2{}_{,\xi}-2pp_\eta\left|\Psi_3\right|^2{}_{,\eta}-2ip^2\left(\Psi_{3,\eta}\overline{\Psi}_{3,\xi}-\Psi_{3,\xi}\overline{\Psi}_{3,\eta}\right) \\ & \left.-2ipp_\xi\left(\Psi_3\overline{\Psi}_{3,\eta}-\Psi_{3,\eta}\overline{\Psi}_3\right)-2ipp_\eta\left(\Psi_{3,\xi}\overline{\Psi}_3-\Psi_3\overline{\Psi}_{3,\xi}\right)\right] \\ & +\frac{9216}{pr^5}\left|\Psi_3\right|^2\left[p\left(\Psi_{3,\sigma}\overline{\Psi}_3+\Psi_3\overline{\Psi}_{3,\sigma}\right)-2\left|\Psi_3\right|^2 p_\sigma\right]. \end{aligned} \tag{3.31}$$

which is obviously a real expression which reduces to the previous one if we restrict $p$ to $p=p(\xi)$. Substituting $\Psi_3$ from 3.30 into it is a tedious but straightforward calculation. We find

$$\begin{aligned} J_2 = {} & \frac{(24)^2p^4}{r^{14}}\left(K_\xi^2+K_\eta^2\right)\left[\frac{1}{8}\left(K_\xi^2+K_\eta^2\right)K+p\left(K_{\xi\eta}^2-K_{\xi\xi}K_{\eta\eta}\right)\right] \\ & +\frac{9}{2}\frac{p^4}{r^{13}}\left[\left(K_\xi^2+K_\eta^2\right)^2\right]_{,\sigma}. \end{aligned} \tag{3.32}$$

Unlike the other known invariants, this one contains a $\sigma$ derivative of $K_\xi^2+K_\eta^2$. This means it offers a different type of information about the space-time which it describes.

In fact, one can conclude that while $J_1$ and $J_B$ depend only on the geometry of each individual light cone, the invariant $J_2$ also depends on the rate of change of the geometry from one light cone to another.

# CHAPTER 4

# LINEAR APPROXIMATION

This chapter is concerned with the existence of type III approximate solutions which are bounded and asymptotically flat in the linear approximation. The method we use is different from the classical one.

Any real solution of a Lorentz invariant theory can be transformed, via a complex Lorentz transformation, into a complex solution. When dealing with linear solutions such as the results of a linear approximation to a gravitational field, the real and imaginary parts are, in principle, new solutions of the theory [35], [32]. When the additional requirement of preserving the algebraic type of the original electromagnetic field or gravitational field is considered, a complex Lorentz transformation is insufficient. Recall that the algebraic type of the Weyl tensor is a property of the self-dual and anti-self-dual parts taken separately. Algebraic integrity is maintained between the real and complex solutions if one proceeds by performing a complex Lorentz transformation on the self-dual solution and then adding its complex conjugate to obtain the real part. This keeps the algebraic structure of the Weyl tensor and in particular any null bivectors which are coincident remain coincident. At this point note that with the self dual and anti-self-dual solutions defining the principal null congruence, the geometry of the principal null congruence has not been preserved.

It is known that a linear approximation to self-dual Robinson-Trautman solutions can produce solutions with twisting rays. However, this method has its limitations.

First, one cannot get the most general solution by this means simply because the twisting congruence obtained via the complex transformation is not the most general one. Second, the resulting solutions may not be in a surveyable form which makes impossible the comparison with the known ones.

In any case, we carry out this procedure explicitly to obtain the most general solution built around what might be described as a generalized Kerr congruence. In addition, this solution is studied by means of an invariant constructed by saturating its first order Bel-Robinson tensor with a time-like vector field which is constant with respect to the zeroth order.

## 4.1 Self-dual Robinson-Trautman spaces

We can complexify the Robinson-Trautman metric, in a purely formal way, by taking the coordinates $\rho$, $\sigma$, $\zeta$, $\widetilde{\zeta}$ to be independent complex variables and the defining functions $m(\sigma)$ and $p(\sigma, \zeta, \widetilde{\zeta})$ to be complex. We are looking for restrictions on the metric functions such that the Weyl tensor becomes self-dual (this means that its anti-self-dual part will be zero). The first obvious condition is

$$m = 0 \tag{4.1}$$

since the function $m$ comes up in both self-dual and anti-self-dual parts. The remaining components of the Weyl tensor can be written in terms of the complex function

$$J = \frac{p_{\zeta\zeta}}{p} \tag{4.2}$$

as

$$^{+}C_{klmn} = \frac{p^3}{\rho^2} J_{\overline{\zeta}}(M_{kl}N_{mn} + N_{kl}M_{mn}) + \frac{1}{\rho^2}\left[\rho p^2 J_\sigma - \left(p^4 J_{\overline{\zeta}}\right)_\zeta\right] N_{kl}N_{mn}. \tag{4.3}$$

The conditions for self-duality are now

$$\widetilde{J}_\zeta = 0 \quad and \tag{4.4}$$

$$\widetilde{J}\sigma = 0 \tag{4.5}$$

or

$$\widetilde{J} = \widetilde{J}(\widetilde{\zeta}) \tag{4.6}$$

This last condition may be strengthen, via a coordinate transformation (see [24]), to

$$\widetilde{J} = 0 \tag{4.7}$$

so that

$$p = A(\sigma, \zeta) + \widetilde{\zeta} B(\sigma, \zeta). \tag{4.8}$$

where $A$ and $B$ are arbitrary functions of $\zeta$ and $\sigma$. The line element is then given by

$$ds^2 = 2d\rho d\sigma + \left(K - 2\rho \frac{p_\sigma}{p}\right) d\sigma^2 - \frac{2\rho^2}{p^2} d\zeta d\widetilde{\zeta} \tag{4.9}$$

$$K = 2(AB_\zeta - A_\zeta B), \qquad p = A + \widetilde{\zeta} B \tag{4.10}$$

The Weyl tensor is

$$C_{klmn} = \frac{p}{2\rho^2} K_\zeta (M_{kl} N_{mn} + N_{kl} M_{mn}) - \frac{1}{2\rho^2} (p^2 K_\zeta)_\zeta N_{kl} N_{mn} + \frac{p^2}{\rho} (p^{-1} p_{\zeta\zeta})_\sigma N_{kl} N_{mn} \tag{4.11}$$

where

$$N = \frac{\rho}{p} d\sigma \wedge d\zeta, \tag{4.12}$$

$$M = d\sigma \wedge d\rho - \frac{\rho^2}{p^2} d\zeta \wedge d\overline{\zeta}. \tag{4.13}$$

## 4.2 Change of coordinates

It is worth remarking that, at least in a special case, one can put this line element into the standard form for one with twisting rays. The special case is defined by $p = A(\zeta)+\widetilde{\zeta}B(\zeta)$. The transformation

$$\begin{aligned}\rho &= r - i\Sigma \\ \zeta &= z \\ \widetilde{\zeta} &= \frac{r+ia}{r-ia}\widetilde{z}.\end{aligned} \tag{4.14}$$

where $\Sigma = a\frac{A(z)-\widetilde{z}B(z)}{P}$ and $P = A(z) + \widetilde{z}B(z)$ transforms the metric 4.9 into

$$ds'^2 = 2\lambda\nu - 2\mu\widetilde{\mu} \tag{4.15}$$

where

$$\lambda = d\sigma + 2Ldz \tag{4.16}$$

$$\mu = P^{-1}(r - i\Sigma)dz, \tag{4.17}$$

$$\widetilde{\mu} = \ P^{-1}(r + i\Sigma)d\widetilde{z}, \tag{4.18}$$

$$\nu = dr + i\left(\Sigma_z dz - \Sigma_{\widetilde{z}} d\widetilde{z}\right) + \frac{1}{2}K\lambda. \tag{4.19}$$

and $L = \frac{ai\widetilde{z}}{P^2}$.

In the new coordinates, the bivectors $N$ and $M$ are given by

$$N_{kl} = \frac{r-ia}{r-i\Sigma}N'_{kl}\,, \tag{4.20}$$

$$M_{kl} = M'_{kl} + \frac{2PKL}{r-i\Sigma}N'_{kl}\,, \tag{4.21}$$

where

$$N' = \lambda \wedge \mu \tag{4.22}$$

$$M' = \lambda \wedge \nu - \mu \wedge \widetilde{\mu}. \tag{4.23}$$

Making this transformation on the flat space line element

$$\underset{0}{ds}^2 = 2d\rho d\sigma + 2kd\sigma^2 - \frac{2\rho^2}{p^2} d\zeta d\widetilde{\zeta} \tag{4.24}$$

$$\underset{0}{p} = 1 + k\zeta\widetilde{\zeta}, \tag{4.25}$$

writing $\overline{z}$ for $\widetilde{z}$ and putting $\sigma = u - ia\frac{z\overline{z}}{\underset{0}{P}}$ with $\underset{0}{P} = 1 + kz\overline{z}$ we get

$$ds^2 = 2\lambda(dr + i\Sigma_z dz - i\Sigma_{\overline{z}} d\overline{z} + k\lambda) - \frac{2(r^2+\Sigma^2)}{\underset{0}{P}^2} dz d\overline{z} \tag{4.26}$$

where $\lambda = du + ia\underset{0}{P}^{-2}(\overline{z}dz - zd\overline{z})$ and $\Sigma = a\frac{1-kz\widetilde{z}}{1+kz\widetilde{z}}$, $a$ being a real parameter.

## 4.3 Linear Approximation

We now examine perturbations of the line element 4.9 by writing

$$p = \underset{0}{p} + \varepsilon \underset{1}{p} \tag{4.27}$$

and working to an accuracy of the first order in $\varepsilon$. It is convenient to take

$$\underset{1}{p} = \alpha_\zeta + \beta\overline{\zeta} + k\overline{\zeta}(\zeta\alpha_\zeta - 2\alpha), \tag{4.28}$$

where $\alpha = \alpha(\sigma, \zeta)$ and $\beta = \beta(\sigma, \zeta)$ are arbitrary functions.

The Weyl is given by

$$\underset{0}{C}_{klmn} = 0, \tag{4.29}$$

$$\begin{aligned}\underset{1}{C}_{klmn} = {} & \frac{1}{\rho^2}\underset{0}{p}\Phi(M_{kl}N_{mn} + N_{kl}M_{mn}) \\ & - \left[\frac{1}{\rho^2}\left(\underset{0}{p}^2\Phi\right)_\zeta - \frac{1}{\rho}\underset{0}{p}\left(\Psi\underset{0}{p} + \Phi_\sigma\widetilde{\zeta}\right)\right] N_{kl}N_{mn}\end{aligned} \tag{4.30}$$

where $\Phi = \beta_{\zeta\zeta}$ and $\Psi = \alpha_{\zeta\zeta\zeta\sigma}$.

Next we transform to the coordinates $u$, $r$, $z, \overline{z}$, restrict ourselves to the space-time in which $u$ and $r$ are real, $z$ and $\overline{z}$ are complex conjugates and we consider the line element $ds^2 = \underset{0}{ds}^2 + \varepsilon \underset{1}{ds}^2 + \varepsilon \underset{1}{d\overline{s}}^2$. Then, using our transformation of bivectors and our expression for $C$ in the linear approximation we get

$$C_{klmn} = \varepsilon \left[ X(M_{kl}N_{mn} + N_{kl}M_{mn}) + YN_{kl}N_{mn} \right] + c.c. \tag{4.31}$$

where

$$X = \frac{P}{(r-ia)(r-i\Sigma)}\Phi\,, \tag{4.32}$$

$$Y = -\frac{P}{(r-ia)^2}\left[\frac{2k\overline{z}(r+ia)}{r-i\Sigma}\Phi + P\Phi_z - \Psi P(r-i\Sigma) - \Phi_\sigma(r+ia)\overline{z}\right]. \tag{4.33}$$

The Bel-Robinson tensor can then be written as

$$\begin{aligned} \frac{1}{2}P_{abcd} &= {}^{+}\underset{1}{C}_{amnc}\,{}^{-}\underset{1}{C}_{b}{}^{mn}{}_{d} \\ &= 4\left|\Gamma\right|^2 \left[\lambda_{(a}\lambda_b\lambda_c\nu_{d)} + 3\lambda_{(a}\lambda_b\mu_c\overline{\mu}_{d)}\right] \\ &+ 4\Gamma\overline{\Delta}\lambda_{(a}\lambda_b\lambda_c\overline{\mu}_{d)} \\ &+ 4\overline{\Gamma}\Delta\lambda_{(a}\lambda_b\lambda_c\mu_{d)} \\ &+ \left|\Delta\right|^2 \lambda_a\lambda_b\lambda_c\lambda_d \end{aligned} \tag{4.34}$$

where

$$\Gamma = X\frac{r-ia}{r-i\Sigma}\,, \tag{4.35}$$

$$\Delta = \frac{2XPKL(r-ia)}{(r-i\Sigma)^2} + Y\left(\frac{r-ia}{r-i\Sigma}\right)^2. \tag{4.36}$$

## 4.4 The generality of the solution obtained

We saw earlier that, with our choices of $\underset{0}{p}$ and $\underset{1}{p}$ given by 4.27 and 4.28, the first approximation of the Weyl tensor is (see 4.30)

$$\begin{aligned} \underset{1}{C}_{klmn} &= \frac{1}{\rho^2}\underset{0}{p}\Phi(M_{kl}N_{mn} + N_{kl}M_{mn}) \\ &\quad - \left[\frac{1}{\rho^2}\left(\underset{0}{p}^2\Phi\right)_\zeta - \frac{1}{\rho}\underset{0}{p}\left(\Psi\underset{0}{p} + \Phi_\sigma\widetilde{\zeta}\right)\right] N_{kl}N_{mn}. \end{aligned} \tag{4.37}$$

In this section we show that this is the most general self-dual type III Weyl tensor that one can have for a Robinson-Trautman metric. To do that we take any linear combination of two *fundamental Weyl tensors* and see what degree of freedom Bianchi identities allow.

The first of the two fundamental tensors (denoted by $III_{klmn}$) is obtained by taking $\alpha = 0$ and $\beta_{\zeta\zeta} = 1$ in the expression 4.37 above. We find

$$III_{klmn} = \frac{1}{\rho^2}\underset{0}{p}(M_{kl}N_{mn} + N_{kl}M_{mn}) - \frac{2}{\rho^2}\underset{0}{p}\underset{0}{p},_{\zeta} N_{kl}N_{mn}. \tag{4.38}$$

The second one (denoted by $N_{klmn}$) is obtained by taking $\alpha_{\zeta\zeta\zeta\sigma} = 1$ and $\beta = 0$ in the same expression 4.37. We have

$$N_{klmn} = \frac{1}{\rho}\underset{0}{p}^2 N_{kl}N_{mn}. \tag{4.39}$$

Any Weyl tensor of the type we are interested in can be written as a linear combination of $III_{klmn}$ and $N_{klmn}$ and can be written in terms of two arbitrary functions $\Phi$ and $\Upsilon$ as

$$C_{klmn} = \Phi III_{klmn} + \Upsilon N_{klmn}. \tag{4.40}$$

The Bianchi identities for a self-dual Weyl tensor simplify to (a proof is included at the end of the section)

$${}^{+}C_{klmn;}{}^{k} = 0 \tag{4.41}$$

and since $III_{klmn}$ and $N_{klmn}$ already satisfy it, equation (4.41) reduces to

$$\Phi,^{k} III_{klmn} + \Upsilon,^{k} N_{klmn} = 0 \tag{4.42}$$

or

$$\frac{1}{\rho^2}\underset{0}{p}\Phi,^{k}(M_{kl}N_{mn} + N_{kl}M_{mn}) + \left[-\frac{2}{\rho^2}\underset{0}{p}\underset{0}{p},_{\zeta}\Phi,^{k} + \frac{1}{\rho}\underset{0}{p}^2\Upsilon,^{k}\right] N_{kl}N_{mn} = 0. \tag{4.43}$$

Contracting by $M^{mn}$ and using that $M_{mn}M^{mn} = -4$ and $N_{mn}M^{mn} = 0$ we find $\Phi,^{k} N_{kl} = 0$ which means

$$\Phi = \Phi(\sigma, \zeta). \tag{4.44}$$

Using this last relation, equation (4.41) becomes

$$\frac{1}{\rho}\Phi,^{k} M_{kl} + \underset{0}{p}\Upsilon,^{k} N_{kl} = 0. \tag{4.45}$$

which reduces further to the following system of equations

$$\underset{0}{p}^{2}\Upsilon,_{\bar{\zeta}} = \Phi,_{\sigma} \tag{4.46}$$

$$\Phi,_{\zeta} = \rho^{2}\Upsilon,_{\rho}\,. \tag{4.47}$$

This is solved by

$$\Upsilon = \Upsilon(\sigma, \zeta) = -\frac{1}{\rho}\Phi,_{\zeta} + \frac{\bar{\zeta}\Phi,_{\sigma}}{\underset{0}{p}} - \Psi(\sigma, \zeta). \tag{4.48}$$

Equations (4.44) and (4.48) represent the highest degree of freedom that one can have for a type III self-dual Weyl tensor for a Robinson-Trautman metric; and they give exactly the coefficients of $\underset{1}{C}_{klmn}$ from 4.37 which means we have obtained the most general type III first order solution built around Kerr's congruence.

**Proof.** The Bianchi's Identities $R_{ab[cd;e]} = 0$ can be written using the Hodge dual as

$$R^{*abcd}{}_{;d} = 0. \tag{4.49}$$

Using the fact that $R_{ab} = 0$ we find

$$C^{*abcd}{}_{;d} = 0 \tag{4.50}$$

or

$$^{*}C^{abcd}{}_{;d} = 0 \tag{4.51}$$

since the left and the right duals for a Weyl tensor coincide. Taking the dual of this last relation and remarking that the dual commutes with the covariant derivative we find that

$$^{**}C^{abcd}{}_{;d} = -C^{abcd}{}_{;d} = 0. \tag{4.52}$$

Equations (4.51) and (4.52) imply now that

$$^{+}C^{abcd}{}_{;d} = 0. \tag{4.53}$$

■

## 4.5 An auxiliary metric and the gravitational density

To study the gravitational field one can look at the differential invariants obtained from the Weyl tensor. However, as we pointed out early in the text, this method is not very practical in showing regularity of space-time; and, if there exist another method of evaluating the gravitational field, the other method should be preferred. In this particular case, the preferred method is the evaluation of the sum of the squares of the Weyl tensor using a positive definite auxiliary metric.

Let

$$\gamma_{ab} = 2t_a t_b - t_r t^r g_{ab} \tag{4.54}$$

where $t$ is a timelike (unit) vector. We have (see [27])

$$^{+}C_{abcd}\,^{-}C_{rstu}\gamma^{ar}\gamma^{bs}\gamma^{ct}\gamma^{du} = (t_r t^r)^2\, P_{abcd}t^a t^b t^c t^d. \tag{4.55}$$

This expression usually depends on $t$ and is of no interest, but in the linear approximation we can take $t$ to be constant with respect to the background (Kerr metric in this case). The easiest way to find such a vector field is to go to Cartesian coordinates. This is accomplished by the transformation given by Robinson et al. (see [24])

$$U = u + \frac{rz\overline{z}}{P} \tag{4.56}$$

$$V = \frac{r}{P} + ku \tag{4.57}$$

$$Z = \frac{r - ai}{P} z \tag{4.58}$$

where the capital letters represent the Minkowski coordinates. Any constant timelike one form, i.e.

$$c_0 dU + c_1 dV + c_2 dZ + \overline{c}_2 d\overline{Z} \tag{4.59}$$

$c_i = const.(i = 1, 2, 3)$ such that $c_0 c_1 - c_2 \overline{c}_2 > 0$, represents a constant timelike one-form in Kerr coordinates.

## 4.6 Discussion

(a) For $k = 0$ a constant tetrad is

$$\widehat{\nu} = \nu \tag{4.60}$$

$$\widehat{\mu} = \mu + z\nu \tag{4.61}$$

$$\widehat{\lambda} = \lambda + \overline{z}\mu + z\overline{\mu} + z\overline{z}\nu. \tag{4.62}$$

A convenient timelike one form is $\tau = \widehat{\lambda} + \widehat{\nu}$; let $t$ be the vector field associated with it. We have

$$\frac{1}{2} P_{abcd} t^a t^b t^c t^d = \left\{ 4 \left|\Gamma\right|^2 + \left|4\overline{z}\Gamma - (1 + z\overline{z})\Delta\right|^2 \right\} (1 + z\overline{z})^2, \tag{4.63}$$

and

$$\Gamma = \frac{\Phi}{(r - ia)^2}, \tag{4.64}$$

$$\Delta = Y|_{k=0}$$
$$= -\frac{1}{(r - ia)^2}\left[\Phi_z - \Psi(r - ia) - \Phi_\sigma(r + ia)\overline{z}\right]. \tag{4.65}$$

One can get nonsingular and asymptotically flat solutions by setting $\Phi = \frac{1}{(\sigma - i)^n}$ with $n > 1$ and $\Psi = \frac{1}{(\sigma - i)^m}$ with $m > 1$.

(b) If $k > 0$, a constant timelike one form is $\tau = k\lambda + \nu$. We have

$$\frac{1}{2}P_{abcd}t^a t^b t^c t^d = 4k\left|\Gamma\right|^2 + \left|\Delta\right|^2. \tag{4.66}$$

For $a = 0$ we have

$$\Gamma = \frac{P}{r^2}\Phi, \tag{4.67}$$

$$\Delta = Y|_{a=0}$$
$$= -\frac{P}{r^2}\left[2k\overline{z}\Phi + P\Phi_z - \Psi P r - \Phi_\sigma r\overline{z}\right] \tag{4.68}$$

The situation is similar to the null case, directional singularities being unavoidable.

# CHAPTER 5

# SECOND APPROXIMATION

In the previous chapter we have obtained pure radiation approximate solutions which are regular in the linear approximation. We want to extend this search to the second order. However, the method applied there cannot be continued because, in the second order, the theory cannot be treated as a Lorentz invariant one. Anything beyond linear approximation has to be done the classical way, i.e. by putting approximations for the main functions $p$ and $L$ into the line element of the general type III space-time. However the previous calculations are very useful in finding $\underset{1}{L}$ ; rather then starting over and solving the field equations for the first approximation, we use the solutions we already found and continue those to the second order. For simplicity, we work in the $p = 1$ gauge.

The solution obtained here is a partial one; that is to say we don't have an expression for $\underset{2}{L}$ but for its first derivative with respect to the zeroth approximation $\underset{0}{\partial}\underset{2}{L}$. There is still some useful information that one can extract from it. We show that the invariant of first order that we obtained earlier is regular for these solutions by showing the regularity of $\underset{2}{\Psi}_3$. We also calculate and show regularity for $\underset{2}{\Psi}_4$.

The search for singularities is limited; the expression for the second order invariant that we have cannot be calculated without the calculation of the full tetrad which we do not have.

## 5.1 Preliminaries

We write down power series for the main functions $p$ and $L$ of the metric in terms of a dimensionless small parameter $\varepsilon$ as follows

$$p = \underset{0}{p} + \varepsilon \underset{1}{p} + \varepsilon^2 \underset{2}{p} + .... \tag{5.1}$$

$$L = \underset{0}{L} + \varepsilon \underset{1}{L} + \varepsilon^2 \underset{2}{L} + .... \quad . \tag{5.2}$$

Obviously we can write such expressions for all functions of the space-times defined in Section 1.6. It is particularly interesting what happens with the $\partial$ operator defined in the same section. It can be written as

$$\partial = \underset{0}{\partial} + \varepsilon \underset{1}{\partial} + \varepsilon^2 \underset{2}{\partial} + .... \tag{5.3}$$

where

$$\underset{0}{\partial} = \partial_\zeta - \underset{0}{L}\partial_u \tag{5.4}$$

$$\underset{i}{\partial} = -\underset{i}{L}\partial_u \quad \forall i \geq 1. \tag{5.5}$$

For the zeroth approximation we take as before

$$\underset{0}{L} = i\overline{\zeta} \tag{5.6}$$

$$\underset{0}{p} = 1 + k\zeta\overline{\zeta} \tag{5.7}$$

initially. However the solutions found in Chapter 4 are regular for $k = 0$ hence, for continuation, we are going to put $\underset{0}{p} = 1$ (and also use $p = 1$ gauge which will imply $\underset{i}{p} = 0$ for any $i \geq 1$) later on.

For later convenience it is useful to know zeroth order derivatives of $\sigma = u - i\zeta\overline{\zeta}$ :

$$\underset{0}{\partial}\sigma = -2i\overline{\zeta} \tag{5.8}$$

$$\underset{0}{\overline{\partial}}\sigma = 0. \tag{5.9}$$

## 5.2 First order approximation

The most general Weyl tensor associated with the linear approximation of a type III solution built around Kerr's congruence was previously shown to be

$$\begin{aligned} \underset{1}{C}_{klmn} &= \frac{1}{\rho^2}\underset{0}{p}\Phi_{\sigma\sigma}(M_{kl}N_{mn}+N_{kl}M_{mn}) \\ &\quad - \left[-2\underset{0}{p}\frac{k\overline{\zeta}}{\rho^2}\Phi_{\sigma\sigma}+\underset{0}{p}^2\frac{1}{\rho}\Upsilon\right]N_{kl}N_{mn} \end{aligned} \tag{5.10}$$

where $\Phi_{\sigma\sigma}=\Phi_{\sigma\sigma}(\sigma,\zeta)$ and $\Upsilon=-\frac{1}{\rho}\Phi_{\zeta\sigma\sigma}+\frac{\overline{\zeta}\Phi_{\sigma\sigma\sigma}}{\underset{0}{p}}+\Psi_{\sigma\sigma\sigma}$, $\Psi=\Psi(\zeta,\sigma)$. From the expression 1.54 of a general self-dual Weyl tensor we see that

$$\underset{1}{\Psi}_3 = \frac{1}{\rho^2}\underset{0}{p}\Phi_{\sigma\sigma}, \tag{5.11}$$

$$\underset{1}{\Psi}_4 = \frac{1}{\rho}\left(\frac{\overline{\zeta}\Phi_{\sigma\sigma\sigma}}{\underset{0}{p}}+\Psi_{\sigma\sigma\sigma}\right)+O\left(\frac{1}{\rho^2}\right). \tag{5.12}$$

Comparing these with the first approximation to the general expressions 1.115 and 1.116 for $\Psi_3$ and $\Psi_4$ one finds the following equations for $\underset{1}{I}$

$$\underset{0}{\overline{\partial}}\underset{1}{I} = \underset{0}{p}^{-2}\Phi_{\sigma\sigma} \tag{5.13}$$

$$\partial_{\sigma\sigma}\underset{1}{\overline{I}} = \frac{\overline{\zeta}\Phi_{\sigma\sigma\sigma}}{\underset{0}{p}}+\Psi_{\sigma\sigma\sigma}. \tag{5.14}$$

Integrating the second expression twice with respect to $\sigma$ and checking with the first one, one finds

$$\underset{1}{\overline{I}} = \frac{\overline{\zeta}\Phi_{\sigma\sigma}}{\underset{0}{p}}+\Psi_{\sigma\sigma}. \tag{5.15}$$

Putting approximations to the first order in the definition of $I$ found in [8]

$$I=\overline{\partial}\left(\overline{\partial}\ln p-\overline{L}_{,u}\right)+\left(\overline{\partial}\ln p-\overline{L}_{,u}\right)^2 \tag{5.16}$$

one finds

$$\underset{1}{\overline{I}} = \left(\underset{0}{p}^{-2}\underset{0}{\partial}\rho\right)_{,u} \tag{5.17}$$

$$\underset{1}{L} = \underset{0}{\partial}\gamma - \rho\underset{0}{p}^{-2} \tag{5.18}$$

where $\ln p = \ln \underset{0}{p} + \varepsilon\gamma_{,u}$ .

The last two expressions and 5.15 are greatly simplified in the $p = 1$ gauge which is going to be used from now on. They become

$$\underset{1}{\overline{I}} = -\underset{0}{\partial}\underset{1}{L}_{,u} \tag{5.19}$$

$$\underset{1}{\overline{I}} = \overline{\zeta}\Phi_{\sigma\sigma} + \Psi_{\sigma\sigma}. \tag{5.20}$$

For the special case $\Phi = \Phi(\sigma)$ and $\Psi = \Psi(\sigma)$ we can integrate the last equation and find

$$\underset{1}{L} = \frac{1}{2i}\Phi + \frac{1}{2i\overline{\zeta}}\Psi. \tag{5.21}$$

Next we remark that these solutions can be written as superpositions of solutions with $\Phi \neq 0$, $\Psi = 0$ and solutions with $\Phi = 0$, $\Psi \neq 0$. The former are solutions of type $N$ and it has been shown elsewhere (see [11]) that they can be regularly continued to the second order of approximation. Therefore, we are going to investigate only solutions with $\Phi \neq 0$, $\Psi = 0$.

Recall now that this first order approximation was shown in Chapter 4 to be regular for a function $\Phi = \frac{2i}{(\sigma - i)^n}$ with $n > 1$ for example[1]. Hence a quite general type III solution regular to the first order is given by

$$L = i\overline{\zeta} + \frac{\varepsilon}{(\sigma - i)^n}, \quad n > 1 \tag{5.22}$$

$$p = 1. \tag{5.23}$$

[1] $n > 0$ is enough for the regularity of the first approximation but not for the second one.

## 5.3 Second order approximation

To calculate $\underset{2}{L}$ we use the real field equation

$$\left(\partial\partial\overline{\partial L}\right)_{,u} = -\left(\partial L\right)_{,u}\left(\overline{\partial L}\right)_{,u}. \tag{5.24}$$

To the second order this gives

$$\left(\underset{0\,0\,0\,2}{\partial\partial\overline{\partial L}} + \underset{0\,1\,0\,1}{\partial\partial\overline{\partial L}} + \underset{1\,0\,0\,1}{\partial\partial\overline{\partial L}} + \underset{0\,0\,1\,1}{\partial\partial\overline{\partial L}}\right)_{,u} = -\left(\underset{0\,1}{\overline{\partial L}_{,u}}\right)\left(\underset{0\,1}{\partial L_{,u}}\right) \tag{5.25}$$

which, although complicated, is a linear equation in $\underset{2}{L}$. However, we were not able to integrate it completely in order to obtain its exact solution.

We are going to investigate the coefficients of the Weyl tensor for regularity. A singularity in any of them would show up as a singularity in a differential invariant, hence a singularity of the space-time. However, the regularity of both of them means only a good choice of the coordinate system.

### 5.3.1 Regularity of $\Psi_3$

We need to calculate $\Psi_3$ to the second order or, since

$$\Psi_3 = -\frac{1}{\rho^2}\partial I, \tag{5.26}$$

$\partial I$ to the second order. We have

$$\underset{2}{\partial I} = \underset{1\,1}{\partial I} + \underset{0\,2}{\partial I} \tag{5.27}$$

and while $\underset{1\,1}{\partial I}$ is obviously regular from the expression of $\underset{1}{I}$, the same is not true for $\underset{0\,2}{\partial I}$. Remark that

$$\underset{0\,2}{\overline{\partial I}} = -\underset{0\,0\,2}{\overline{\partial}\partial L_{,u}} \tag{5.28}$$

expression which is fairly easy to calculate from 5.25. We have

$$\underset{0}{\partial}\underset{0}{\partial}\underset{0}{\overline{\partial}}\underset{2}{L}_{,u} = \frac{4\zeta\overline{\zeta}n^2\,(n+1)\,(n+2)}{|\sigma-i|^{2n+2}\,(\overline{\sigma}+i)^2} - \frac{4in^2(n+1)}{|\sigma-i|^{2n+2}\,(\overline{\sigma}+i)} - \frac{4in\,(n+1)\,(n+2)}{|\sigma-i|^{2n}\,(\overline{\sigma}+i)^3} \tag{5.29}$$

which can be integrated to

$$\begin{aligned} \underset{0}{\partial}\underset{0}{\overline{\partial}}\underset{2}{L}_{,u} \quad = \quad & -\frac{2i\zeta n(n+1)(n+2)}{|\sigma-i|^{2n}\,(\overline{\sigma}+i)^3} - \frac{n(n+1)(n+2)}{(n-1)}\frac{1}{\overline{\zeta}}\frac{1}{|\sigma-i|^{2n-2}\,(\overline{\sigma}+i)^4} \\ & -\frac{2n(n+1)}{\overline{\zeta}\,|\sigma-i|^{2n}\,(\overline{\sigma}+i)^2} + A(\overline{\zeta},\overline{\sigma}). \end{aligned} \tag{5.30}$$

At first look this expression seems singular because of the $\frac{1}{\overline{\zeta}}$ factor. However, the function of integration $A$ may be chosen so that the singularity is removed (for example take $A = 3n^2\frac{n+1}{n-1}\frac{1}{\overline{\zeta}}\frac{1}{(\overline{\sigma}+i)^{2n+2}}$).

The remaining field equation

$$Im\partial\partial\overline{\partial}\overline{L} = 0 \tag{5.31}$$

does not impose any condition on the integration function $A$. As a matter a fact, to the second order, we have

$$\left(\underset{2}{\partial\partial\overline{\partial}L}\right)_{,u} = \frac{4\zeta\overline{\zeta}n^2\,(n+1)^2}{|\sigma|^{2n+4}}. \tag{5.32}$$

The expression in the right side is a real continuous function with respect to $u$, $\zeta$, $\overline{\zeta}$ hence integrable; the field equation then says that the integration function is real.

An immediate conclusion of the regularity of $\Psi_3$ to the second order is that the first order invariant $J_1$ described in section 3.1 and given by

$$J_1 = 6^2\,|4\rho\Psi_3|^4 \tag{5.33}$$

is also regular to the second order.

### 5.3.2 Regularity of $\Psi_4$

The expression of $\Psi_4$ for a type III space-time was shown in [36] to be

$$\Psi_4 = \psi_4^0 \rho + Z_1 \rho^2 + \frac{1}{2} Z_2 \rho^2 \tag{5.34}$$

where

$$Z_1 = 2\left(\eth - 4L,_u\right)\left(\psi_3^0\right) \tag{5.35}$$

$$Z_2 = 4N\psi_3^0 \tag{5.36}$$

and $N = 2i\left(\eth - 3L,_u\right)\left(\overline{\eth} L - \eth \overline{L}\right)$.

To the second order, they have the following expressions

$$\underset{2}{Z_1} = 2\left(\underset{0}{\eth}\underset{2}{\psi_3^0} + \underset{1}{\eth}\underset{1}{\psi_3^0} - 4\underset{1}{L},_u \underset{1}{\psi_3^0}\right) \tag{5.37}$$

$$\underset{2}{Z_2} = 0 \tag{5.38}$$

and the regularity follows from the expressions for $\underset{1}{\Psi_3}$ and $\underset{2}{\Psi_3}$.

The regularity of $\Psi_4$ depends then only on the regularity of $\partial_u I$ to the second order or, since

$$\underset{2}{\overline{I}} = \underset{1}{L},_u^2 + \underset{1}{L}\underset{1}{L},_{uu} - \underset{0}{\eth}\underset{2}{L},_u \,, \tag{5.39}$$

on the regularity of $\underset{0}{\eth}\underset{2}{L},_{uu}$. To calculate this last expression, we integrate 5.30 and find

$$\overline{\underset{0}{\eth}\underset{2}{L},_u} = -\frac{n(n+1)(n+2)}{n-1}\frac{\zeta}{\overline{\zeta}}\frac{1}{|\sigma - i|^{2n-2}(\overline{\sigma}+i)^4} \tag{5.40}$$

$$+\frac{in(n+1)}{n-1}\frac{1}{\overline{\zeta}^2}\frac{1}{|\sigma - i|^{2n-2}(\overline{\sigma}+i)^3} \tag{5.41}$$

$$+A(\overline{\zeta},\overline{\sigma})\zeta + B(\overline{\zeta},\overline{\sigma}). \tag{5.42}$$

Remark that since $\left|\frac{\zeta}{\overline{\zeta}}\right| = 1$, both the first and the third terms are regular while the function $B$ can be chosen to make the second one regular (for example take $B(\overline{\zeta},\overline{\sigma}) = -\frac{in(n+1)}{n-1}\frac{1}{\overline{\zeta}^2}\frac{1}{(\overline{\sigma}+i)^{2n+1}}$).

APPENDIX

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