

# Analytic continuation of perturbative solutions of acoustic 1D wave equation by means of Padé approximants

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Communicated by R.P. Gilbert

(Received 3 September 2005; in final form 29 June 2006)

The forward scattering series is an important and useful tool in constructing perturbative solutions to wave equation and understanding their relationship to their non-perturbative counterparts. When it converges, the series describes the total wavefield everywhere in a given medium as propagations in a reference medium and interactions with point scatterers. The method can be viewed as constructing a mapping between non-perturbative solutions of wave events and their volume point scatterer description. This mapping was shown to be required by the recently developed techniques for inverse problems based on the inverse scattering series with applications to seismic exploration (Weglein, A.B., Gasparotto, F.A., Carvalho, P.M. and Stolt, R.H., 1997, An inverse scattering series method for attenuating multiples in seismic reflection data. *Geophysics*, **62**, 1975–1989, Weglein, A.B., Araujo, F.V., Carvalho, P.M., Stolt, R.H., Matson, K.H., Coates, R., Corrigan, D., Foster, D.J., Shaw, S.A. and Zhang, H., 2003, Inverse scattering series and seismic exploration. *Topical Review Inverse Problems*, **19**, R27–R83). The forward scattering series for a 1D acoustic medium and a normal incidence plane wave was shown in Matson, K.H., 1996, The relationship between scattering theory and the primaries and multiples of reflection seismic data. *J. Seis. Expl.*, **5**, 63–78 to converge for a ratio less than  $\sqrt{2}$  between the reference and the actual velocity. Same restricted convergence was obtained in Innanen, K.H., 2003, Methods for the treatment of acoustic and absorbtive/dispersive wavefield measurements, PhD Thesis, Department of Earth and Ocean Sciences, University of British Columbia, Vancouver, Canada for a visco-acoustic medium with or without dispersion. In this article, we propose an explanation for this divergence and an extension of the method able to construct the solution of the 1D wave equation for any velocity contrast between the actual and the reference medium for both acoustic and visco-acoustic cases. The method involves the analytic continuation of the forward scattering solution by computing a certain sequence of Padé approximants to the partial sums of the forward scattering series.

*Keywords:* Forward scattering series; Acoustic wavefields; Analytic continuation; Padé approximants

*2000 Mathematics Subject Classifications:* 35J05; 35R12; 35P25; 34E10

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## 1. Introduction

Scattering theory is a powerful method for analyzing wave propagation in a given medium. As a form of perturbation theory, it relates the propagation of the wave in that medium with the propagation of the wave in a reference medium and a perturbation operator which describes the difference between the two media. The forward problem is to construct the actual wavefield everywhere given the reference wavefield and the perturbation operator; the inverse problem is to construct the perturbation operator (and hence the unknown medium) given the reference wavefield everywhere and the actual wavefield on a measurement surface outside the unknown medium (collected data). The relation between these three quantities is nonlinear and does not have a closed form representation in either the forward or the inverse problem. This relationship takes the form of a series which, when convergent, constructs the actual wavefield and the perturbation operator respectively.

The inverse scattering series is presently the only non-linear, direct inversion method for the multi-dimensional, acoustic or elastic wave equation. Originated by Jost and Kohn [5] and later developed by Moses [11], its convergence has been studied, among others, by Prosser [14], who concluded that the convergence of the forward series is necessary for the convergence of the inverse series. At an even more fundamental level, one can see from the construction of the inverse series (see e.g. [5,11,17]) that the convergence of the forward is necessary for the very existence of the inverse scattering series.

In the '90's, Weglein and collaborators developed the subseries method (for a history and description see [17]) which consists in identifying task specific subseries in the full series, with targeted usefulness and better convergence properties than the whole series. The process of identifying these subseries was shown in [16] to require the forward scattering description of wave events.

The forward scattering series for a 1D wavefield propagating in acoustic media was studied by Matson [7] who showed that convergence occurs for a ratio less than  $\sqrt{2}$  between the reference and the actual velocity. The study was later extended by Matson [8] and Nita *et al.* [12] to a 2D wavefield propagating in a vertically varying acoustic medium; they showed that the forward series only converges for either limited velocity contrast or limited incidence angle respectively. Innanen [4] studied the forward scattering series for a 1D wavefield propagating in a visco-acoustic medium and found, consistently with the previous results, that the series converges only for a limited contrast between the actual and the reference medium.

In this article we further study the convergence of the forward scattering series for a 1D wavefield propagating in an acoustic medium. We show that the divergence of the series is due to the iterative process of constructing it. Furthermore, we propose an extension of this method by using a sequence of Padè approximants to the partial sums of the series, to extend the convergence to any velocity contrast. It is reasonable to approximate a singular function (the solution to the wave equation) with a sequence of Padè approximants rather than with a Taylor series (the forward scattering series). The approximants are themselves singular and, in this case, it turns out that their singularities are all located along the branch cut of the function which is approximated. Hence the sequence can be interpreted as building up the function everywhere by mimicking its branch cut singularity with poles located along it. The result can easily

be extended to the visco-acoustic case to show that the sequence of Padé approximants converges to the exact result for arbitrary velocity contrast between the reference and the actual medium.

The sections in this article are structured as follows. In section 2, we present the mathematical development for the forward scattering series from a 1D acoustic wave equation; section 3 discusses the convergence of the constructed series; section 4 introduces Padé approximants and section 5 discusses their connection with continued fractions; in section 6 we show the convergence of the Padé approximants for the 1D case for any velocity contrast for this specific model. Section 7 discusses the construction of the reflection coefficient as a function of the velocity perturbation; some conclusions and ideas for future work are presented in section 7.

## 2. Forward scattering series for a 1D model

The differential equation describing a 1D wave propagating in an acoustic, constant density medium is

$$\left[ \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z)} \frac{\partial^2}{\partial t^2} \right] P(z|z_s; t) = \delta(z - z_s) \delta(t), \quad (1)$$

where  $P(z|z_s; t)$  represents the pressure field at the point  $z$  at time  $t$  due to a source at  $z_s$  which exploded at time  $t=0$ . The above equation (1) assumes that the source signature has been deconvolved. Fourier transforming it with respect to time yields

$$\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2(z)} \right] P(z|z_s; \omega) = \delta(z - z_s), \quad (2)$$

where  $\omega$  is the temporal frequency. The velocity in the actual medium  $c(z)$  can be characterized in terms of a reference velocity  $c_0$ , chosen to be constant, and a perturbation  $\alpha(z)$  such that

$$\frac{1}{c^2(z)} = \frac{1}{c_0^2} [1 - \alpha(z)] \quad (3)$$

or

$$\alpha(z) = 1 - \frac{c_0^2}{c^2(z)}. \quad (4)$$

Substituting equation (3) into equation (2) gives

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_0^2} \right) P(z|z_s; \omega) = \delta(z - z_s) + \frac{\omega^2}{c_0^2} \alpha(z) P(z|z_s; \omega). \quad (5)$$

Note that, putting  $\alpha(z) = 0$  in equation (5) leads to the the equation for wave propagation in an homogeneous medium, in this case the reference medium. The solution would represent only the direct arrival from the source located at  $z_s$  to the receiver located at  $z$ . Hence the right-hand side of equation (5) can be interpreted as describing an effective source made of two terms: the delta term describing the source exploding in the reference medium and the term containing  $\alpha(z)$  describing the contribution of the inhomogeneity to the total wavefield, i.e., the scattered field.

To find a solution to equation (5), consider the causal, free space Green's function  $P_0(z|z_s; \omega)$  satisfying

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_0^2}\right)P_0(z|z_s; \omega) = \delta(z - z_s). \quad (6)$$

Then, an integral equation corresponding to equation (5) and its physical boundary conditions is [15]

$$P(z|z_s; \omega) = P_0(z|z_s; \omega) + \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z') P(z'|z_s; \omega) dz'. \quad (7)$$

This last equation is the Lippmann–Schwinger equation, the fundamental equation of scattering theory. It represents the wavefield everywhere in an inhomogeneous medium as the sum of the wavefield in a reference medium and the scattered field due to a perturbation.

Equation (7) can be expanded in an infinite series by repeatedly substituting  $P(z|z_s; \omega)$  from the left into the right-hand side to obtain the forward scattering series

$$\begin{aligned} P(z|z_s; \omega) &= P_0(z|z_s; \omega) + \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z') P_0(z'|z_s; \omega) dz' \\ &+ \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z') \int_{-\infty}^{\infty} P_0(z'|z''; \omega) \frac{\omega^2}{c_0^2} \alpha(z'') P_0(z''|z_s; \omega) dz'' \\ &+ \dots \end{aligned} \quad (8)$$

When this series converges, it constructs the scattered field as a series of terms formed with propagations in the reference medium  $P_0$  and interactions with the inhomogeneity  $\alpha(z)$ . In this article, we analyze the convergence of this series for the simplest, one interface case.

Following Matson [7], consider two semi-infinite half spaces with constant wave velocities  $c_0$  and  $c_1$  and the interface between them located at  $z = z_1$ . The reference medium is chosen as a homogeneous whole space with velocity  $c_0$ . The velocity perturbation  $\alpha$  can be then written as

$$\alpha(z) = \alpha_0 H(z - z_1), \quad (9)$$

where  $\alpha_0 = 1 - c_0^2/c_1^2$ . Assume that the plane source is located at  $z_s = 0$ . The one dimensional Green's function representing the plane-wave propagation in the reference medium is (see e.g. [10])

$$P_0(z|z_s = 0; \omega) = \frac{e^{ik|z|}}{2ik}, \quad (10)$$

where we denoted  $k = (\omega/c_0)$ . After putting the expressions (9) and (10) into equation (8) and solving the integrals, we obtain the total wavefield everywhere above the interface, i.e. direct arrival and reflection, to be [7]

$$P(z < z_1 | 0; \omega) = \frac{e^{ik(2z_1 - z)}}{ik} \left( \frac{1}{2} + \frac{1}{8}\alpha_0 + \frac{1}{16}\alpha_0^2 + \frac{5}{128}\alpha_0^3 + \frac{7}{256}\alpha_0^4 + \frac{21}{1024}\alpha_0^5 + \dots \right). \quad (11)$$

Nita *et al.* [12] showed that the expression in the parenthesis is a Taylor series which had been recognized in [7] as the Taylor series for

$$\frac{1 - \sqrt{1 - \alpha_0}}{\alpha_0} \quad (12)$$

calculated at  $\alpha_0 = 0$ . It is easy to see that the expression (12) is

$$\frac{1 - \sqrt{1 - \alpha_0}}{\alpha_0} = \frac{1}{2} \left( 1 + \frac{c_1 - c_0}{c_1 + c_0} \right); \quad (13)$$

hence, when it converges, the series constructs the reflected wavefield  $P^R$  to be

$$P^R(z < z_1 | 0; \omega) = \frac{e^{ik(2z_1 - z)}}{2ik} \frac{c_1 - c_0}{c_1 + c_0}. \quad (14)$$

This is the exact expression of the reflected wavefield for a normal incidence plane wave reflected from an interface in a 1D medium obtained through non-perturbative methods (see e.g. [1]).

### 3. The convergence of the forward scattering series

The derivation of the reflected wavefield in section 2 depends on the convergence of the Taylor series expansion for the function  $\sqrt{1 - \alpha_0}$  shown in figure 1. The ratio test shows that the series converges when  $|\alpha_0| < 1$ , or, in terms of velocities, when

$$\frac{c_0}{c_1} < \sqrt{2}. \quad (15)$$

For this simple model, this case covers all lower-over-higher velocity and limited contrast higher-over-lower velocity situations. The lack of a similar limitation in the

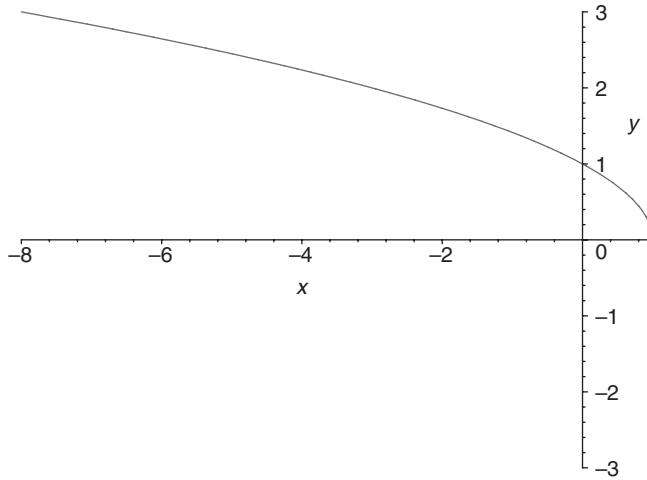


Figure 1. The graph of  $\sqrt{1-x}$ .

non-perturbative case is an indication that this is not an issue specific to the wave theory but it is rather a fault in the method. In this section, we take a closer look at the construction of the series and propose a reason for this divergence.

Equation (8) is obtained by repeatedly substituting  $P(z|z_s; \omega)$  from the left hand side of equation (7) into the right-hand side of the same equation. At any step of this iteration, the new equation obtained is an exact implicit solution for the total wavefield everywhere. For example, the equations

$$P(z|z_s; \omega) = P_0(z|z_s; \omega) + \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z) P(z'|z_s; \omega) dz', \quad (16)$$

$$P(z|z_s; \omega) = P_0(z|z_s; \omega) + \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z) P_0(z'|z_s; \omega) dz' \\ + \int_{-\infty}^{\infty} P_0(z|z'; \omega) \frac{\omega^2}{c_0^2} \alpha(z') \int_{-\infty}^{\infty} P_0(z'|z''; \omega) \frac{\omega^2}{c_0^2} \alpha(z'') P(z''|z_s; \omega) dz'' \quad (17)$$

and so on, are all exact equations for  $P$ . The role of the last term on the right-hand side is to balance the beginning of the series and to bring that expression to be equal to the actual wavefield. For example, if the first terms in the series are large, the series begins increasing with a divergent trend. However, in this case, the last term would counteract this behavior through a similar increase in magnitude but with an opposite sign hence making the equation exact. When the number of terms becomes infinite, that last terms is dismissed and what remains is called the forward scattering series. In this case, if the first terms in the series are large and the series has a divergent trend, then the series will diverge.

The following simple algebraic example illustrates the same phenomenon at a lower, and easier to understand, scale. Let

$$f(x) = x + 2f(x) \quad (18)$$

be an equation in which the unknown is the real function  $f$ . One can check very easily that  $f(x) = -x$  is the solution of (18). However, rather than calculating the solution, we want to obtain it as a series in the variable  $x$  through an iteration procedure similar to the one involved in the construction of the forward scattering series. For this purpose, we substitute  $f(x)$  from the left-hand side into the right-hand side of (18). At first step we obtain

$$f(x) = x + 2x + 4f(x), \quad (19)$$

which is an exact equation, at second step

$$f(x) = x + 2x + 4x + 8f(x), \quad (20)$$

which is, again, exact, and so on. If we continue the procedure one could conclude that the solution to equation (18) is

$$f(x) = x \sum_{n=0}^{\infty} 2^n; \quad (21)$$

however, this series is divergent for all  $x$  except  $x=0$ . Hence the function is not constructible through this procedure anywhere except  $x=0$ . Note that a similar example yielding an everywhere convergent series could have been obtained for the functional equation  $f(x) = x + (1/2)f(x)$ .

One can see from this simple example that, although the sum in the iterations starts to increase by adding more terms, the term containing  $f(x)$  at the end counteracts the divergent trend through a similar increase in magnitude and an opposite sign. For each fixed  $x$ , although adding more terms only drives the series away from the value of the function, the coefficient of  $f(x)$  also increases and balances that error. Throwing away that last term by writing the series expression in (21) leaves a divergent series behind instead of an exact result.

The conclusion for the forward scattering series is the following: the series begins as a Taylor series and ends in a term involving the actual wavefield which assures having an exact result for any velocity contrast. In writing the full series, that last term is omitted which results in a convergent series for a small variable, i.e.  $|\alpha_0| < 1$ , and a divergent one for a large variable, i.e.,  $|\alpha_0| > 1$ .

This behavior suggests the use of Padé approximations to the partial sums in the forward scattering series. For a given such partial sum, say  $P_1 + P_2 + \dots + P_n$ , the Taylor series of the corresponding Padé approximant has, by definition, exactly the same first  $n$  terms but a different ending than the full Taylor series. It turns out that this different ending is able to make up for the dismissal of the term containing the actual wavefield in the series, and hence leading to a sequence of Padé approximants which is convergent for any velocity contrast for this given model. What's also remarkable is that each Padé approximant can be constructed from the corresponding partial sum in the forward scattering series only. This, and the connection with the continued fractions, will be discussed in the following sections.

#### 4. Padé approximants

A Padé approximant  $P_M^N(x)$ , can be regarded as a rational function approximation to the  $(M + N + 1)$ -th order partial sum of a given power series  $\sum a_n x^n$ . Such a mathematical object can be written as

$$P_M^N(x) = \frac{\sum_{n=0}^N A_n x^n}{\sum_{n=0}^M B_n x^n} \quad (22)$$

where we can take  $B_0 = 1$  without any loss of generality. The remaining  $(M + N + 1)$  coefficients  $A_0, A_1, \dots, A_N, B_1, B_2, \dots, B_M$  are chosen so that the first  $M + N + 1$  terms in the Taylor expansion of  $P_M^N(x)$  match the first  $M + N + 1$  terms of the power series  $\sum a_n x^n$ . The resulting rational function is called a Padé approximant of order  $(M, N)$ .

Constructing the Padé approximants  $P_M^N(x)$  is often very useful. For a discussion of their properties and applications, see e.g. [2]. The convergence of Padé approximants to functions containing branch points has been discussed in [13]. Their application to critical phenomena and scattering theory has been presented in [2,3,9]. If  $\sum a_n x^n$  is a power series representation of the function  $f(x)$ , then, in many instances,  $P_M^N(x)$  converges to  $f(x)$  as  $M, N \rightarrow \infty$ , even if  $\sum a_n x^n$  is a divergent series. An important feature to notice is that the full power series of a function is not needed to construct a Padé approximant, only the first  $M + N + 1$  terms. Next, we consider the power series in the expansion (11) of the function (12) and show how Padé approximants for its partial sums can be constructed. For reasons that will be apparent later, we are going to consider only the special sequence made of the following Padé approximants:  $P_0^0, P_1^0, P_1^1, P_2^1, P_2^2, P_3^2, \dots$ . After denoting  $\alpha_0 = x$ , it can be easily seen that

$$P_0^0(x) = \frac{1}{2} \quad (23)$$

$$P_1^0(x) = \frac{1/2}{1 - (1/4x)} \quad (24)$$

$$P_1^1(x) = \frac{1/2 - 1/8x}{1 - (1/2x)} \quad (25)$$

$$P_2^1(x) = \frac{(1/2) - (1/4x)}{1 - (3/4x) - (1/16x^2)} \quad (26)$$

and so on. Notice that, as aforementioned, the Taylor series of  $P_M^N$  calculated at  $x = 0$  coincides with the first  $(M + N + 1)$  terms in the power series in (11); for example, the Taylor series of  $P_2^1$  at  $x = 0$ ,

$$\text{Taylor}(P_2^1)(x) = \frac{1}{2} + \frac{1}{8}x + \frac{1}{16}x^2 + \frac{5}{128}x^3 + \frac{13}{512}x^4 + \frac{17}{1024}x^5 + O(x^6), \quad (27)$$

coincides with the power series in (11) to the fourth term. It can be seen, from the figures 1 and 2, that, for this specific case, any given Padé approximant represents a better approximation, to the function to be constructed, than its corresponding partial sum in the forward scattering series. In the following sections, we will discuss the connection of the Padé approximants to continued fractions and prove that this

sequence is convergent everywhere in the whole complex  $x$ -plane except for the cut from  $-\infty$  to 1 along the real axis; in contrast, the forward scattering series converges only for real values of  $x$  such  $|x| < 1$ . Note that the convergence of the sequence of Padé approximants for any real  $x < 1$  (or, by definition, any  $\alpha_0 < 1$ ) covers all velocity contrasts in the acoustic case. Moreover, the convergence anywhere in the complex plane except the cut from  $x = 1$  to  $-\infty$  along the real axis, also covers, for this given model, all velocity contrasts in the case of complex actual velocities, i.e., the visco-acoustic case.

### 5. Continued fractions and Padé approximants

A continued fraction is an infinite sequence of fractions whose  $(N + 1)$ th member  $F_N(z)$  has the form

$$F_N(x) = \frac{c_0}{1 + \frac{c_1 x}{1 + \frac{c_2 x}{1 + \dots \frac{c_{N-1} x}{1 + c_N x}}}}. \tag{28}$$

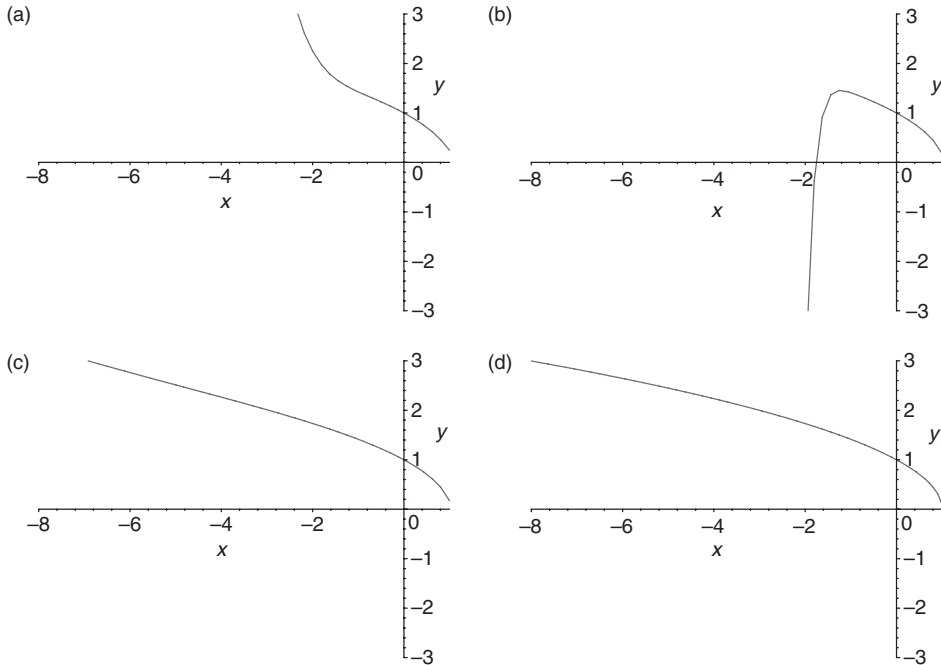


Figure 2. Comparison of partial sums in the Taylor series (above) and Padé approximants (below) for the function  $\sqrt{1-x}$  pictured in figure 1. Note that any given Padé approximant represents a better approximation to the function than its corresponding Taylor partial sum. (a) The graph of the first 6 terms in the Taylor series for  $\sqrt{1-x}$ , (b) the graph of the first 11 terms in the Taylor series for  $\sqrt{1-x}$ , (c) the graph of the padé approximant  $P_3^2$ . Recall that the Taylor series of  $P_3^2$  coincides with the Taylor series for  $\sqrt{1-x}$  up to the 6th term, (d) the graph of the padé approximant  $P_5^2$ . Recall that the Taylor series of  $P_5^2$  coincides with the Taylor series for  $\sqrt{1-x}$  up to the 11th term.

A power series representation for a certain function can be recasted in a continued fractions format; in this case the coefficients  $c_n$  are determined by expanding  $F_N(x)$  in a Taylor series and comparing the coefficients with those in the power series to be summed. The procedure resembles Padé summation since, as before, only algebraic operations are required for the calculation.

The sequence of Padé approximants of the form considered in previous section, i.e.  $P_0^0, P_1^0, P_1^1, P_2^1, P_2^2, P_3^2, \dots$ , is called normal if every member exists and no two members are identically equal. It can be shown that if this Padé sequence is normal, then the  $(N+1)$ th term has the continued fraction representation (25) with the coefficients  $c_n$  being the same for every term of the sequence. In other words,  $P_{M+1}^M(x)$ , for  $M > 0$ , is obtained from  $P_M^M(x)$  by simply replacing  $c_{Nx}$  by  $c_{Nx}/(1 + c_{N+1}x)$  where  $N = 2M$ , and  $P_{M+1}^{M+1}(x)$ , for  $M \geq 0$ , is obtained from  $P_M^M(x)$  by replacing  $c_{Nx}$  by  $c_{Nx}/1 + c_{N+1}x$  where  $N = 2M + 1$ . In contrast to the Padé approximants representation as a ratio of two polynomials, where, for each higher rank, every coefficient in the rational fraction must be recomputed, in this representation, only one new coefficient need be computed as we go from one member to the next.

The continued fraction (28) can be evaluated in the obvious direct way. However, if the function  $F_N(x)$  is written as

$$F_N(x) = \frac{R_N(x)}{S_N(x)}, \quad (29)$$

for  $N = 1, 2, \dots$ , then it can be shown by induction that the functions  $R_N$  and  $S_N$  satisfy the recurrence relations

$$R_{N+1}(x) = R_N(x) + c_{N+1}xR_{N-1}(x), \quad (30)$$

$$S_{N+1}(x) = S_N(x) + c_{N+1}xS_{N-1}(x). \quad (31)$$

These relations can be used to study the convergence of the Padé sequence  $F_N(x)$ . To see this, multiply equations (30) and (31) by  $S_N(x)$  and  $R_N(x)$  respectively and subtract the resulting equations to obtain

$$R_{N+1}(x)S_N(x) - R_N(x)S_{N+1}(x) = -c_{N+1}x(R_N(x)S_{N-1}(x) - R_{N-1}(x)S_N(x)), \quad (32)$$

for  $N \geq 1$ . This equation and the initial conditions  $S_0 = 1$ ,  $S_1 = 1 + c_1x$ ,  $R_0 = c_0$  and  $R_1 = c_0$ , give

$$R_N(x)S_{N-1}(x) - R_{N-1}(x)S_N(x) = c_0c_1 \dots c_N (-x)^N. \quad (33)$$

Dividing this last equation by  $S_N S_{N-1}$  and recalling that  $F_N = R_N/S_N$  we find

$$F_N(x) - F_{N-1}(x) = \frac{c_0c_1 \dots c_N (-x)^N}{S_N(x)S_{N-1}(x)}. \quad (34)$$

The last equation shows that the behavior of the denominators  $S_N(x)$  for large  $N$  determines the convergence of the sequence  $F_N(x)$ . This behavior can be found from

an asymptotic analysis of the difference equation (27) if the behavior of the continued fraction coefficients  $c_N$  for large  $N$  is known.

## 6. The convergence of Padé approximants for the 1D model

In this section, we apply the notions defined above to our specific 1D example. The function that we are going to discuss is chosen for simplicity to be [compare with equation (12)]

$$f(x) = \frac{\sqrt{1-x} - 1}{2x}. \quad (35)$$

Since the difference between the limit of the convergent forward scattering series and  $f(x)$  consists only in algebraic operations, we are going to refer to  $f(x)$  as the limit of the forward scattering series.

As shown above, the forward scattering series for the 1D acoustic model, converges to  $f(x)$  for any complex  $x$  such that  $|x| < 1$ . In the following, we are going to show that the sequence of Padé approximants  $F_N(x) = (P_N^N(x), P_{N+1}^N(x))$ , constructed from the partial sums of the Taylor series for  $f(x)$ , is convergent to  $f(x)$  everywhere in the complex domain  $\mathbb{D} = \{x \in \mathbb{C} : |\arg(1-x)| < \pi\}$ . The proof will follow the following steps. First, we are going to use Vitali's theorem to show that the sequence of Padé approximants  $F_N(x)$  converges to an analytic function in  $\mathbb{D}$ . Second, we are going to show that this analytic function coincides with  $f(x)$  on the negative real axis; analyticity will then imply that the limit function equals  $f(x)$  everywhere on  $\mathbb{D}$ .

Vitali's theorem states that (see e.g. [6]) for a given sequence of functions which are analytic in a simply connected region  $\mathbb{D}$  and uniformly bounded over every finite closed domain lying within  $\mathbb{D}$ , and which converges at an infinite set of points having at least one limit point interior to  $\mathbb{D}$ , then the sequence converges throughout  $\mathbb{D}$  uniformly in finite closed domains lying within  $\mathbb{D}$  to an analytic function.

To begin verifying the assumptions in Vitali's theorem, first note that  $f(x)$  satisfies the algebraic equation

$$f(x) = \frac{-1/4}{1 + xf(x)} \quad (36)$$

and hence a continued fraction expansion is easily obtained by substituting the above equation into itself:

$$f(x) = \frac{-1/4}{1 - \frac{x/4}{1 - xf(x)}} = \frac{-1/4}{1 - \frac{x/4}{1 - \frac{x/4}{1 - xf(x)}}} = \dots, \quad (37)$$

Thus, the continued fraction coefficients of  $f(x)$  are  $c_N = -1/4$  for all  $N \geq 0$ . The denominators  $S_N(x)$  satisfy [see equation (31)]

$$S_{N+1} = S_N - \frac{1}{4}xS_{N-1} \quad (38)$$

and with  $S_0 = 1$  and  $S_1 = 1 - x/4$ . This is a constant coefficient difference equation which may be solved in a closed form and obtain

$$S_N(x) = \frac{1}{\sqrt{1-x}} \left[ \left( \frac{1 + \sqrt{1-x}}{2} \right)^{N+2} - \left( \frac{1 - \sqrt{1-x}}{2} \right)^{N+2} \right]. \quad (39)$$

It is easy to see that, for  $x \in \mathbb{D}$ , we have  $|1 + \sqrt{1-x}| > |1 - \sqrt{1-x}|$  and hence  $S_N > 0$  on the real axis from  $-\infty$  to 1 and  $S_N \neq 0$  on  $\mathbb{D}$ . This implies that  $F_N(x)$  has exactly  $N$  simple poles all located along the positive real axis from 1 to  $\infty$  and hence  $F_N(x)$  is analytic on  $\mathbb{D}$ ; moreover, it gives an important information about the structure and behavior of the Padè approximants. What the method is trying to accomplish is construct a sequence of functions which converges to a complex function containing a branch cut ( $\sqrt{1-x}$ ). The Taylor series cannot handle discontinuities hence it diverges. However, a Padè approximant could do a better job since it contains singularities itself (the zeros of the polynomial at the denominator). What we have just shown is that all these singularities are located along the branch cut of the original function. The number of poles in a Padè approximant increases with the order; hence the sequence can be interpreted as building up the branch cut from simple poles giving a more accurate description of the function  $f(x)$  than its Taylor series representation.

To show boundedness over finite closed domains lying in  $\mathbb{D}$  we notice that a simple fractions decomposition of, say,  $P_N^N$  is

$$P_N^N(x) = a_0 + x \sum_{q=1}^N \frac{C_q}{1 - D_q x} \quad (40)$$

with  $C_q \geq 0$  and  $D_q \geq 0$  for all  $q$ . We also have that  $\sum_{q=1}^N C_q = a_1$  and that  $\sum_{q=1}^N C_q/D_q < a_0$ . Then one can show that

$$|P_N^N(x)| \leq a_0 + a_1|x| \quad (41)$$

for  $x$  in  $\mathbb{D}$  such that  $\mathbf{Re}(x) \leq 0$  and

$$|P_N^N(x)| \leq a_0 + a_0 \frac{|z|}{\mathbf{Im}(z)} \quad (42)$$

for  $x$  in  $\mathbb{D}$  such that  $\mathbf{Re}(x) > 0$ . A similar argument can be employed to show boundedness on finite closed domains in  $\mathbb{D}$  for any  $P_{N+1}^N(x)$ .

The next step is to show that the sequence  $F_N(x)$  converges for an infinite set of points having a limit point interior to  $\mathbb{D}$ . This will be accomplished by showing that the subsequences  $P_N^N(x)$  and  $P_{N+1}^N(x)$  are monotonically increasing and decreasing, respectively, with  $N$ , and that

$$P_N^N(x) \leq f(x) \leq P_{N+1}^N(x). \quad (43)$$

To verify the monotony of the two Padé subsequences we make use of two equations relating the numerator  $R_N(x)$  and the denominator  $S_N(x)$  of the continued fraction  $F_N(x) = R_N(x)/S_N(x)$ :

$$R_N(x)S_{N-1}(x) - R_{N-1}(x)S_N(x) = c_0 c_1 \dots c_N (-x)^N \quad (44)$$

$$R_{N+1}(x)S_{N-1}(x) - R_{N-1}(x)S_{N+1}(x) = R_N(x)S_{N-1}(x) - R_{N-1}(x)S_N(x). \quad (45)$$

The first of these equations is equation (33) and the second follows from equations (30) and (31). Using the fact that  $c_N = -1/4$  for all  $N$  and that  $x < 0$ , we see that

$$\text{sgn}(R_N(x)S_{N-1}(x) - R_{N-1}(x)S_N(x)) = (-1)^{N+1} \quad (46)$$

and hence we also have

$$\text{sgn}(R_{N+1}(x)S_{N-1}(x) - R_{N-1}(x)S_{N+1}(x)) = (-1)^{N+1}. \quad (47)$$

Since  $S_N(x)$  is positive for all  $N$  [see equation (31) and the definitions of  $S_0$  and  $S_1$ ], by dividing the last equation by  $S_{N-1}(x)S_{N+1}(x)$  we find that

$$\text{sgn}(F_{N+1}(x) - F_{N-1}) = (-1)^{N+1} \quad (48)$$

or, in other words,

$$F_{2N+1}(x) < F_{2N-1}(x) \quad (49)$$

$$F_{2N+2}(x) > F_{2N}(x). \quad (50)$$

Noticing that  $F_{2N+1}(x) = P_{N+1}^N(x)$  and  $F_{2N}(x) = P_N^N(x)$  proves the monotony of the two Padé subsequences.

To prove the second statement first note that the function  $f(x)$  can be obtained from the continued fraction  $F_N(x)$ , given by (28), by replacing  $c_N x$  by  $c_N x/(1 + xf(x))$ . Then, if we perform this substitution in equations (30) and (31), written with the index lowered by 1, we find

$$R(x) = R_{N-1}(x) + \frac{c_N x}{1 + xf(x)} R_{N-2}(x), \quad (51)$$

$$S(x) = S_{N-1}(x) + \frac{c_N x}{1 + xf(x)} S_{N-2}(x). \quad (52)$$

Multiply the first of these two equations by  $S_{N-1}(x)$  and the second by  $R_{N-1}(x)$  and subtract the resultant equations to find

$$R(x)S_{N-1}(x) - S(x)R_{N-1}(x) = \frac{c_N x}{1 + xf(x)} (R_{N-2}(x)S_{N-1}(x) - R_{N-1}(x)S_{N-2}(x)). \quad (53)$$

The right-hand side is given by equation (33) hence we obtain

$$R(x)S_{N-1}(x) - S(x)R_{N-1}(x) = \frac{c_0 c_1 \dots c_N (-x)^N}{1 + xf(x)}. \quad (54)$$

As before, since the sign of the right-hand side is  $(-1)^{N+1}$  we have

$$\text{sgn}(R(x)S_{N-1}(x) - S(x)R_{N-1}(x)) = (-1)^{N+1}. \quad (55)$$

In addition to the fact that  $S_N(x) > 0$  for all  $N$ , from equation (52), we also have  $S(x) > 0$  and hence when we divide the last equation by the product  $S_N(x)S(x)$  we find

$$\text{sgn}\left(\frac{R(x)}{S(x)} - \frac{R_{N-1}(x)}{S_{N-1}(x)}\right) = (-1)^{N+1}, \quad (56)$$

or,

$$\text{sgn}(f(x) - F_{N-1}(x)) = (-1)^{N+1}. \quad (57)$$

Since  $F_{2N}(x) = P_N^N(x)$  and  $F_{2N+1}(x) = P_{N+1}^N(x)$ , the last equation translates into

$$P_N^N(x) \leq f(x) \leq P_{N+1}^N(x). \quad (58)$$

This last step in Vitali's theorem proves the convergence of the Padé sequence  $F_N(x)$  on the negative real axis. It also shows that the analytic function that the sequence converges to everywhere on  $\mathbb{D}$ , equals  $f(x)$  for all real  $x < 0$ . Analyticity then implies that  $F_N(x)$  converges to  $f(x)$  everywhere on  $\mathbb{D}$ .

## 7. Modeling the reflection coefficient

In this section, we discuss the construction of the reflection coefficient, for this model, for all possible velocity contrasts. It is not difficult to see [see e.g. equation (13)] that the expression for the reflection coefficient as a function of the perturbation  $\alpha_0$  for this single interface model is

$$R = \frac{2}{\alpha_0} \left( 1 - \frac{1}{2}\alpha_0 - \sqrt{1 - \alpha_0} \right) \quad (59)$$

Note that the reflection coefficient is well defined for any real or complex values  $\alpha_0$ , except along the branch cut singularity located along the real  $\alpha_0$  axis for  $\alpha_0 > 1$ . The real values of  $\alpha_0$  such that  $\alpha_0 < 1$  correspond to all acoustic models while the allowable complex values of  $\alpha_0$  correspond to visco-acoustic models.

Figure 3 shows the ability of the forward scattering series to well approximate the reflection coefficient for real valued perturbations such that  $-1 < \alpha_0 < 1$  or, in terms of velocities,  $c_0/c_1 < \sqrt{2}$ . However, for a large set of acoustic models, the series diverges

and hence it is unable to model the reflection coefficient. In consequence, the series fails to construct the reflected or transmitted wavefield. Moreover, this has implications for multidimensional cases where this divergence translates as inability to model post-critical plane wave reflections [12]. We showed earlier in this paper that the divergence is due to the method of constructing the forward scattering series. We also showed that using a sequence of Padé approximations of the partial sums of the series, one can model the wavefield for any velocity contrast for this 1D case. Figure 4 exemplifies this construction. For comparison purposes, same orders of approximations were used for figures 3 and 4. Notice that, at any step, the sequence of Padé approximants provides a better approximation to the values of the reflection coefficient. Moreover, the method introduced in this article seems to be extremely accurate even at early terms in the sequence and it is able to reproduce both the non-singular and the singular part of the reflection coefficient. We emphasize that the branch cut singularity of the reflection coefficient, located on the real axis

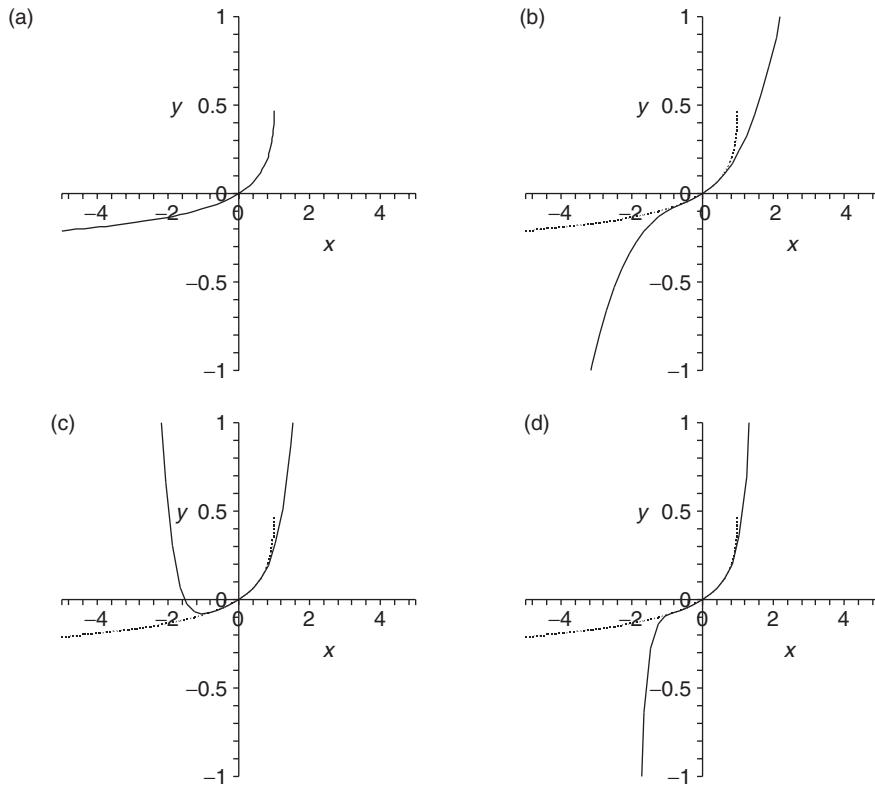


Figure 3. Different order of approximations of the reflection coefficient calculated as a function of the perturbation  $\alpha_0$ . The exact values are shown in figure 3(a) and are given as a reference point in all the other pictures. Figure 3(b), (c) and (d) show different orders of calculations using the forward scattering series. Note that, consistent with the analytical results, the series well approximates the reflection coefficient for perturbations  $-1 < \alpha_0 < 1$  and diverges for all other cases. (a) The exact values of the reflection coefficient as a function of the perturbation  $\alpha_0$ , (b) the fourth order approximant, (c) the seventh order approximant, (d) the tenth order approximant.

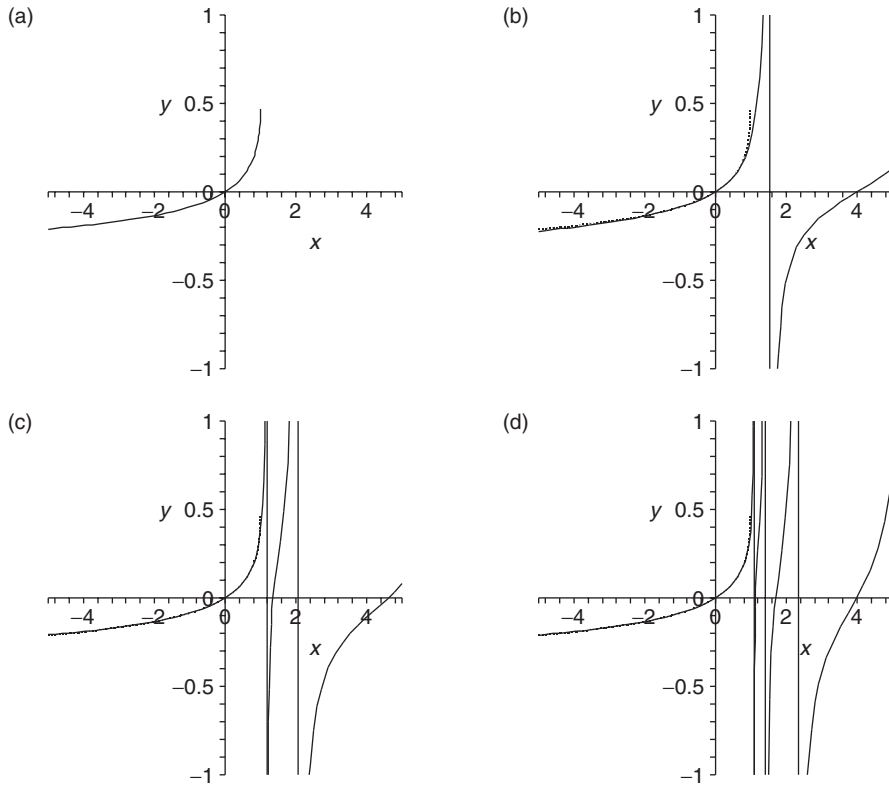


Figure 4. Different order of approximations of the reflection coefficient calculated as a function of the perturbation  $\alpha_0$ . The exact values are shown in figure 4(a) and given as a reference point in all the other pictures. Figure 4(b), (c) and (d) show different orders of calculations using Padé approximants. (a) The exact values of the reflection coefficient as a function of the perturbation  $\alpha$ . (b) the fourth order Padé approximant, (c) the seventh order Padé approximant, (d) the tenth order Padé approximant.

for  $\alpha_0 > 1$ , is built up from discrete poles located along it which become more numerous as the order of the approximants increases. This allows not only the estimation of the reflection coefficient in the acoustic but also the visco-acoustic cases. The positive implications of these items for the multi-dimensional case will be discussed elsewhere.

## 8. Conclusions

In this article, we have presented a detailed analysis of the construction and the convergence of perturbative solutions of the 1D acoustic wave equation using the forward scattering series. The model discussed here contains two half spaces with the interface located at  $z_1$ , with constant wave propagation velocities  $c_0$  and  $c_1$  and with no density contrast. The source, located in the first medium at  $z=0$ , emits a vertically downward propagating wave which hits the interface and produces a reflection. When the series converges this reflection is constructed as an infinite sum of terms representing

propagations in the reference medium and interactions with point scatterers which define the actual medium.

The forward scattering series for this model converges for a ratio less than  $\sqrt{2}$  between the reference and the actual velocity. We have shown that this limitation is due to the iterative method of constructing the series. In addition we have introduced a sequence of Padé approximants which extends the convergence of the solution to any velocity contrast. The discussion also covers the visco-acoustic case where the convergence of the sequence of approximants for any velocity contrast was showed. The final method of constructing a general solution, for this case, using perturbation techniques can hence be described in two steps: first calculate the partial sum in the forward scattering series to the desired order and then calculate the Padé approximant associated with it. At any step the later provides a better approximation to the exact solution than the former. Interestingly enough, only the partial sum of order  $N + M + 1$  is needed for the calculation of the corresponding Padé approximant  $p_M^N$ .

This research is an important step forward in analyzing the perturbative solutions of acoustic wave equation and, more generally, of ordinary and partial differential equations. The results presented here can be extended to the multidimensional wave equation describing propagations in a vertically varying medium. For this type of media, Nita *et al.* [12] showed that the forward scattering series can construct solutions for pre-critical incident plane waves and diverges in other cases. We expect that the method of Padé approximants discussed in this article will be able to extend this construction to post-critical events and even allow the scattering description of more complex wave phenomena (e.g. headwaves). These ideas and the application of this method to model wavefield propagation in more complex media, as well as to solving inverse problems, will be considered in future work.

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