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Mathematical structure of voting paradoxes*

II. Positional voting

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Summary. A theory is developed to explain all positional voting outcomes that can result from a single but arbitrarily chosen profile. This includes all outcomes, paradoxes, and disagreements among positional procedure outcomes as well as all discrepancies in rankings as candidates are dropped or added. The theory explains why each outcome occurs while identifying all illustrating profiles. It is shown how to use this approach to derive properties of methods based on pairwise and positional voting outcomes. Pairwise voting is addressed in the preceding companion paper [15]; the theory for positional methods is developed here.

Keywords and Phrases: Positional voting, Voting paradoxes, Social choice.

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1 Introduction

This paper continues the development of an approach which characterizes, explains, and illustrates all positional and pairwise voting outcomes generated by a single but arbitrarily chosen profile. To recall, positional procedures assign points to alternatives according to how a voter positions them on the ballot. Familiar choices are the plurality vote where a single point is assigned to a voter's top-ranked candidate and zero to all others, and the Borda Count (BC) where

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$n-1, n-2, \ldots, n-n=0$ points are assigned, respectively, to a voter's first, second, ..., nth ranked candidate. The candidates are ranked according to the sum of assigned points. A "pairwise vote" is a majority vote election of two candidates. The preceding paper(Saari [15]) describes the importance and difficulties of this project.

Recall; instead of the traditional axiomatic method, the approach introduced here develops a "coordinate system" for profile space. Profiles from a specified direction – and only these profiles – affect particular classes of procedures. Thus this "profile coordinate system" significantly simplifies the analysis of voting procedures to something resembling elementary vector analysis. The companion paper (Saari [15]) describes the profile coordinate directions for pairwise outcomes and "kernel" profiles (i.e., profiles with no effect upon any pairwise or positional method; see Section 4.1). It remains to characterize the profile directions causing all positional election difficulties over all subsets of candidates; this is done here.

With this full profile coordinate system, we can, for instance, determine, explain and illustrate all changes in positional outcomes caused by adding or dropping candidates. To illustrate how bad the situation can be, consider the subsets of candidates $\{c_1, c_2, \ldots, c_n\}$, $\{c_1, c_2, \ldots, c_{n-1}\}$, ..., $\{c_1, c_2\}$ obtained by dropping a candidate at each stage. Choose a ranking for each subset; select them randomly or to deliberately create a particularly perverse example. For each set, choose a positional procedure – they can agree (e.g., use the plurality vote with each set), or change with the set. With the specified information, a profile exists so that the assigned procedure's election outcome for each subset is the chosen ranking (Saari [8]). Thus, for instance, there is a ten-candidate profile with the plurality ranking $c_2 \succ c_3 \succ c_4 \succ \cdots \succ c_{10} \succ c_1$ even though the same profile's plurality rankings of the other eight subsets agree with $c_1 \succ c_2 \succ c_3 \succ \cdots \succ c_9$. Eight election rankings support c_1 as the candidate of choice, but, by admitting what appears to be an "essentially irrelevant candidate" c_{10}, c_1 falls to the bottom. The combined results of this paper and [15] allow all such examples to be described, illustrated with profiles, and explained.

In another direction, we now can describe, explain, and illustrate all paradoxes where a profile's election rankings radically change with the method. This is a serious issue because, as proved in (Saari [11]), a ten-candidate profile exists (and I indicate in Section 5.5 how to construct one) which generates millions upon millions of different election outcomes with different positional methods. Each alternative "wins" with certain procedures, but is bottom-ranked with others.

So, by using this approach, one way to analyze these types of issues (which reflect central concerns of choice theory) is to determine the components of a specified profile; these components identify all associated paradoxical or positive behavior. Conversely, to create a profile with particularly perverse outcomes, just add the appropriate profile components which cause the outcomes of certain procedures to deviate as wildly as desired from the outcomes of other procedures. Examples are provided to help develop intuition and understanding.

1.1 Explanations

The decomposition relies upon the *Basic profiles* (Section 4.2) where all positional procedures and pairwise votes *share the same election ranking and even the same (normalized) election tally* for all subsets of candidates. No conflict of any kind – caused by dropping candidates or using different methods – occurs on the Basic portion. Consequently, all voting difficulties are caused by profiles which are orthogonal to the space of Basic profiles. These components are called "profile deviations" to reflect that they, and only they, cause outcomes to deviate from the idyllic electoral setting of Basic profiles; they create all possible election oddities. While it will be arguable that the deviation profiles should cause completely tied election outcomes, this is not the case; instead, all election paradoxes and inconsistencies occur when procedures deny this conclusion.

While these comments suggest that no procedure is reliable, this is not the case. A fundamental conclusion is that *the BC outcome for all n-candidates is the unique ranking which avoids all of the indicated problems.* This result follows from the unexpected fact (Saari [15]) that the BC applied to all n candidates is the only procedure which ignores all profile deviations – its ranking is strictly determined by the Basic component. (This assertion does *not* hold for the BC rankings of subsets of candidates (Saari [8, 15]).) Consequently, all differences in the election rankings of other procedures and subsets of candidates are caused by the deviations ignored by the BC ranking of all n candidates. In particular, differences in tallies between other procedures and the BC measure the affect of each profile deviation.

An explanation for an important class of paradoxes is given in Section 8 of Saari [15] and Saari [14]. The idea is that while each voter has complete, strict transitive preferences, a procedure which emphasizes only subsets of candidates loses this crucial rationality information. Then, anonymity (where the outcomes do not depend on the identity of the voters) prevents the procedure from distinguishing between profiles representing a heterogeneous but rational society and those representing a society where voters are only partially transitive. Thus, inconsistent rankings of subsets of candidates – even those ranked with the BC – partially reflects how information about the individual rationality of voters is ignored.

To indentify the source of other profile deviations, suppose after an election leading to the ranking of $A \succ B \succ C \succ D$, it is discovered that each voter marked the ballot completely opposite of what had been intended. It is reasonable to assume that the correct ranking is the reversed $D \succ C \succ B \succ A$. As an example, profile

Number of voters	Ranking	Number of voters	Ranking
1	$A \succ B \succ C \succ D$	1	$D \succ C \succ B \succ A$
2	$A \succ B \succ D \succ C$	2	$C \succ D \succ B \succ A$
3	$A \succ D \succ C \succ B$	3	$B \succ C \succ D \succ A$

has the plurality ranking $A \succ B \succ C \succ D$ with tally 6:3:2:1. But when each ranking is reversed (to find the new tally, compute the number of times each candidate is bottom ranked in this listing), the plurality ranking *remains* $A \succ B \succ C \succ D$ with the identical 6:3:2:1 tally. To explain this phenomenon, notice that because each row has a ranking and its reversal with the same number of voters, anonymity ensures that the profile and its reversal are the same. It is reasonable to expect such pairing of "opposites" to force a cancellation of the voters' votes. But, if a positional procedure cannot recognize this informational symmetry, it provides an outcome different from the natural tie. This captures the general behavior; *all profile deviations reflect some procedure's inability to recognize certain kinds of informational symmetry.*

To use this brief description, return to the above described behavior (from Saari [8]) allowing the rankings for any specified positional methods – even the BC – to change with the subsets of candidates. It turns out that two kinds of deviations are involved where the first (Saari [15]), the Condorcet profiles (Section 4.3), changes the rankings of pairs *and* of all positional methods in an identical manner for each *proper subset* of candidates. (It does not affect positional rankings for the set of all n candidates.) This deviation type reflects how the profile information ignored by procedures weakens the individual rationality assumption. The second type, based on the above "reversal type symmetry," affects all non-BC positional procedures. A consequence is that all non-BC positional procedures admit more kinds of paradoxical outcomes than the BC. Another surprising consequence is that the BC variation of rankings of subsets (from the Condorcet portion) is sufficiently natural that we should worry about choice procedures which do *not* exhibit similar variations in rankings as candidates are added or dropped. These assertions contradict widely accepted beliefs where the last one contradicts a central research objective.

1.2 Outline

After notation and earlier results are briefly reviewed (Section 2), I develop the structure of positional voting methods (Section 3). (In doing so, I describe a surprisingly easy way to geometrically determine and quickly display all possible positional election outcomes admitted by a four-candidate profile.) An interesting aspect of this vector space structure is how it proves that the election outcomes for certain k-candidate positional procedures are uniquely and completely determined by the positional outcomes of certain s-candidate procedures, $s < k$. Whenever this occurs, we must expect, and it is true, that these procedures admit fewer election oddities and more consistency with election relationships. Moreover, the vector space structure makes it easy to identify the procedures exhibiting these consistency properties. There are surprises; some of the new election relationships counter intuition about what can or should occur, so they identify large classes of new paradoxical behavior.

The profile decomposition starts (Section 4) with a brief review of the surprisingly large space of Kernel profiles which have no effect on any positional or pairwise method. I then review the properties of the $(n-1)$ dimensional space of Basic profiles which experience no faults. The first space of deviation profiles is the $(n-1)!/2$ dimensional space of Condorcet differentials mentioned above. To avoid certain undesired properties, a $\binom{n-1}{2}$-dimensional subspace is defined. As these profiles affect only pairwise outcomes, they simplify the analysis.

In the Introduction, I mentioned that a profile can be constructed where different procedures have radically different election outcomes. Indeed, an example constructed by Borda in his seminal paper [2], where the plurality and BC outcomes differ, started the academic study of choice theory. In fact, it is reasonable to believe that most papers on this topic (e.g., see Kelly [5]) worry about how and why procedures can yield different outcomes. We now have an answer; different procedures use different aspects of information about the profile. The prominence of this issue justifies creating a special profile decomposition (Section 5) to handle all possible positional outcomes for the n-candidates with all possible associated pairwise outcomes, to explain the behavior by identifying the different types of information, and to derive new conclusions.

The general setting, which describes what happens to all subsets of candidates, is in Section 6. Applications and implications of the profile decomposition are described in Section 7 where a matrix is given to compute the four-candidate different profile deviations for any profile $\mathbf{p}$. A higher dimensional version of Basic and Condorcet profiles is introduced in Section 8. Most proofs are in Section 9.

2 Preliminaries

The notation is as in Saari [15]. List, in any manner, the $n!$ transitive ways to strictly rank the $n \geq 3$ candidates $\{c_1, c_2, \ldots, c_n\}$. Each ranking defines a *voter type*; a *profile* specifies the number of voters of the jth type, $j = 1, \ldots, n!$. The n candidates define $2^n - (n+1)$ subsets of two or more candidates; list them in some manner as $S_1, S_2, \ldots, S_{2^n-(n+1)}$ where $|S_j|$ is the number of candidates in S_j. To represent the S_j election tally as a vector in $R^{|S_j|}$, assign each $R^{|S_j|}$ axis to a S_j candidate in increasing order of the subscripts. So, for $S_j = \{c_2, c_4, c_5\}$, the $(4, 23, 13) \in R^3$ *vector tally* defines the ranking $c_4 \succ c_5 \succ c_2$ with the 23:13:4 tally.

Denote a S_j positional voting method by *voting vector* $\mathbf{w}^{S_j} = (w_1, w_2, \ldots, w_{|S_j|})$ where $w_1 = 1$, $w_{|S_j|} = 0$, $w_j \geq w_{j+1} \geq 0$ for $j = 1, \ldots, |S_j| - 1$. This normalized choice (of $w_1 = 1$ and $w_{|S_j|} = 0$) simplifies the comparisons of procedures and results. Converting a voting vector into its normalized form is trivial; e.g., a four-candidate subset tallied with $(7, 3, 2, 0)$ has the normalized version $(\frac{7}{7}, \frac{3}{7}, \frac{2}{7}, 0)$. Similarly, if the plurality method is used with $\{c_1, c_3, c_4\}$ and the BC with $\{c_2, c_3, c_4, c_5\}$, then the normalized voting vectors are, respectively, $(1, 0, 0)$ and $(1, \frac{2}{3}, \frac{1}{3}, 0)$. To tally ballots with $\mathbf{w}^{S_j}$, assign w_j points to a voter's jth ranked candidate, $j = 1, \ldots, |S_j|$ and rank the candidates according to the sum of assigned points.

The *system voting vector*, $\mathbf{W}^n = (\mathbf{w}^{S_1}, \dots, \mathbf{w}^{S_{2^n-(n+1)}})$ specifies that $\mathbf{w}^{S_j}$ is used to tally the S_j election, $j = 1, \dots, 2^n - (n+1)$. Let $F(\mathbf{p}, \mathbf{W}^n)$ and $\tilde{F}(\mathbf{p}, \mathbf{W}^n)$ be, respectively, the lists of election tallies and rankings of all subsets of candidates defined by profile **p** with system vector $\mathbf{W}^n$. To illustrate with $S_1 = \{c_1, c_2\}$, $S_2 = \{c_1, c_3\}$, $S_3 = \{c_2, c_3\}$, $S_4 = \{c_1, c_2, c_3\}$, the system vector $\mathbf{W}^3 = [(1,0), (1,0), (1,0), (1,0,0)]$ requires the pairs to be tallied with the majority rule (voting vector (1,0)) and the triplet with the plurality vote (voting vector (1,0,0)). The fifty-voter profile **p** where three voters have preferences $c_1 \succ c_2 \succ c_3$, 24 have $c_3 \succ c_1 \succ c_2$, and 23 have $c_2 \succ c_1 \succ c_3$ defines the election rankings

$$\tilde{F}(\mathbf{p}, \mathbf{W}^3) = [c_1 \succ c_2,\ c_1 \succ c_3,\ c_2 \succ c_3,\ c_3 \succ c_2 \succ c_1] \tag{2.1}$$

which exhibit conflicting pairwise and plurality rankings. All such inconsistencies are characterized and explained.

2.1 Words

The numbers and kinds of admissible election inconsistencies — paradoxes — are staggering. To illustrate, Eq. 2.1 lists the election ranking for each subset of candidates coming from the specified profile — call such a listing a *word* (Saari [12]). Different profiles can define different plurality words so the number and kinds of words measures the complexity and randomness of a procedure. It turns out that the plurality vote admits 351 different words for $n = 3$ candidates and over a *billion* (1,041,048,450) for $n = 4$ candidates. Namely, there are over a billion different ways to list rankings (many involve ties) for the six pairs, four triplets, and the set of all four candidates, and each listing is the sincere plurality election outcome for some profile. These numbers overwhelm any naive belief that the election rankings of the pairs and triplets must agree with that of all four candidates. (If this naive wish were true, only 50 plurality words could occur. Of these, 4! = 24 have no ties, the rest have at least one tie vote.) The following assertion (which concerns all subsets of candidates rather than just the nested sets considered in [8]) demonstrates the severity of the problem.

Theorem 1. *(Saari [12]) For $n \geq 3$ candidates, suppose all subsets of candidates are tallied with the plurality method. For each subset, choose a ranking. As these rankings can be selected randomly, there need not be a relationship among them. There exists a profile so that the voters' sincere plurality ranking of each subset of candidates is the selected one.*[1]

So, there is a profile where its plurality rankings of subsets with an even number of candidates match $c_1 \succ c_2 \succ c_3 \succ \cdots \succ c_n$, but the plurality ranking

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[1] To use Theorem 1 to compute the number of four-candidate plurality words, notice that there are three ways to rank a pair (including ties), thirteen ways to rank a triplet, and 50 ways to rank four candidates. Thus, the total number of words is $3^6 \times (13)^4 \times 50$. Similarly, the number of plurality words – election paradoxes – for five candidates escalates over ten million billion fold to $3^{10} \times (13)^{10} \times (50)^4 \times 630 = 3.21 \times 10^{25}$.

for subsets with an odd number of these candidates is reversed. A more disturbing conclusion is to use a random number generator to select the ranking of each subset and be assured that (Theorem 1) there is a profile where the voters' sincere plurality election ranking of each set agrees with the randomly generated rankings. This is not a ringing endorsement for our standard tool of democracy.

2.2 Other procedures

This plurality electoral nightmare is shared by most methods. The next result uses the fact that $\mathbf{W}^n \in R^{\nu(n)}$, $\nu(n) = 2^{n-1}(n-4) + n + 2$. (The derivation of $\nu(n)$ is in Section 9.) In $R^{\nu(n)}$, an *algebraic set* is a lower dimensional subset representing the zeros of a particular collection of polynomials.

Theorem 2. *(Saari, [12]) With the exception of an algebraic set $\alpha^n \subset R^{\nu(n)}$, all other system vectors in $R^{\nu(n)}$ have the same property as described for the plurality vote in Theorem 1.*

Only the highly exceptional $\mathbf{W}^n \in \alpha^n$ tallying procedures offer consistency in outcomes with election relationships. The α^n entries and election relationships are described.

3 Division of voting vectors

As stated, it will be shown that paradoxes manifest how procedures treat different profile deviations. In turn, each subspace of profile deviations defines a subspace of positional procedures which react to this particular type of profile information. This dual decomposition uses the linearity of $F(\mathbf{p}, \mathbf{W}^n)$ in each variable.

To illustrate F's linearity with respect to the voting vector, suppose a four-candidate election is tallied with both $(5, 2, 1, 0)$ and $(2, 1, 0, 0)$. Because $(9, 4, 1, 0) = 2(2, 1, 0, 0) + (5, 2, 1, 0)$, the $(9, 4, 1, 0)$ election tally is the same as adding twice each candidate's tally from the second election to her tally from the first one. In normalized form, the computation is $(1, \frac{4}{9}, \frac{1}{9}, 0) = \frac{5}{9}(1, \frac{2}{5}, \frac{1}{5}, 0) + \frac{4}{9}(1, \frac{1}{2}, 0, 0)$. Implications follow.

3.1 Procedure hull

A n-candidate voting vector is a convex combination of the $(n-1)$ voting vectors $\{\mathbf{v}_j^n\}_{j=1}^{n-1}$ where $\mathbf{v}_j^n$'s first j components are ones and the rest are zeros. (So, $\mathbf{v}_j^n$ represents the n-candidate election where we vote for j candidates.) Clearly, the convex hull defined by $\{\mathbf{v}_j^n\}_{j=1}^{n-1}$ includes all (normalized) voting vectors for n candidates. Denote this *n-candidate pyramid of voting vectors* by

$$\mathscr{P}^n = \{\, \mathbf{w}^n = \sum_{j=1}^{n-1} \lambda_j \mathbf{v}_j^n \mid \lambda_j \geq 0, \sum_{j=1}^{n-1} \lambda_j = 1\}. \tag{3.1}$$

The $\mathscr{P}^n$ dimension of $(n-2)$ reflects the $n-2$ weights needed to define a n-candidate voting vector. The BC vector $\mathbf{b}^n = \frac{1}{n-1}\sum_{j=1}^{n-1}\mathbf{v}_j^n$ is at the $\mathscr{P}^n$ barycenter.

To illustrate with the ten-voter profile

Number	Preference	Number	Preference
2	$A \succ B \succ C \succ D$	2	$C \succ B \succ D \succ A$
1	$A \succ C \succ D \succ B$	3	$D \succ B \succ C \succ A$
2	$A \succ D \succ C \succ B$		

(3.2)

the $\mathbf{v}_1^4 = (1,0,0,0)$, $\mathbf{v}_2^4 = (1,1,0,0)$, and $\mathbf{v}_3^4 = (1,1,1,0)$ respective tallies are $(5,0,2,3)$ where A is the winner, $(5,7,3,5)$ where B is the winner, and $(5,7,10,8)$ where C is the winner. A $\mathscr{P}^4$ voting vector is expressed as $\sum_{j=1}^{3}\lambda_j\mathbf{v}_j^4$, so its election tally for the profile is $\lambda_1(5,0,2,3)+\lambda_2(5,7,3,5)+\lambda_3(5,7,10,8)$. All election tallies are in the triangle (called the *procedure hull*, Saari [11]) with vertices defined by the three $\{\mathbf{v}_j^4\}_{j=1}^3$ election tallies; e.g., the $\mathbf{b}^4$ outcome of $(5, 4\frac{2}{3}, 5, 5\frac{1}{3})$, with D as the winner, is at the barycenter (where all $\lambda_j = \frac{1}{3}$).

Notice how this innocuous ten-voter profile allows *each candidate to "win" with one of these commonly used procedures.* Similarly, there is a ten-candidate profile $\mathbf{p}$ where its *procedure hull* – the $F(\mathbf{p}, -)$ image of $\mathscr{P}^{10}$ – has over 84 million different election rankings as the tallying procedure changes (Saari [11]). This significantly extends earlier results (e.g., a well known one is in Fishburn [4]) asserting that profiles exist which admit two different outcomes.

3.2 Basis

For reasons that will become clear, it is convenient to represent voting vectors with an orthogonal basis centered at the barycentric point $\mathbf{b}^n$.

Proposition 1. *For $n \geq 3$ candidates, let $\mathbf{E}_j^n$ be the vector with unity in the jth component and zero in all others. All voting vectors $\mathbf{w}^n$ can be expressed as the sum*

$$\mathbf{w}^n = \mathbf{b}^n + \sum_{j=2}^{n-1}\alpha_j\mathbf{E}_j^n \tag{3.3}$$

for appropriate choices of α_j.

Proof. $\mathbf{v}_k^n = \mathbf{b}^n + \sum_{j=2}^{n-1}\alpha_j\mathbf{E}_j^n$ where $\alpha_j = 1 - \frac{n-j}{n-1}$ for $j = 2, \dots k$ and $\alpha_j = -\frac{n-j}{n-1}$ for $j = k+1, \dots, n-1$. Because each $\mathbf{v}_k^n$ has the indicated Eq. 3.3 expression and because each $\mathscr{P}^n$ vector is a convex sum of $\{\mathbf{v}_k^n\}_{k=1}^{n-1}$, all voting vectors have the Eq. 3.3 expression. □

The barycentric location of $\mathbf{b}^n$ creates a natural, orthogonal coordinate representation for voting vectors. To compute the $\mathbf{w}^n = \mathbf{b}^n + \sum_{j=2}^{n-1}\alpha_j\mathbf{E}_j^n$ tally for a profile, first compute the normalized BC outcome. Then, for each j, add α_j times the $\mathbf{E}^n_j$ tally. (The $\mathbf{E}^n_j$ tally for c_i is the number of times voters rank her in jth position.) The importance of this representation comes from the fact (illustrated next) that the $\mathbf{b}^n$ tally is determined by the pairwise votes. Consequently, Eq. 3.3 indicates how and why the tallies for non-BC procedures differ from pairwise outcomes. (See, for instance, Eq. 2.1.)

To see this BC relationship for $n = 3$, note that a voter with preferences $c_1 \succ c_2 \succ c_3$ votes as follows in the three pairwise elections.

Candidates	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$
$\{c_1, c_2\}$	1	0	–
$\{c_1, c_3\}$	1	–	0
$\{c_2, c_3\}$	–	1	0
Total	2	1	0

(3.4)

The sum of votes this voter gives a candidate over all pairs equals what he assigns her in a BC election. Thus (along with neutrality and the fact that each pair is tallied with the same voting vector) a candidate's BC election tally is the sum of the two pairwise election tallies she receives in contests against candidates in the same subset; the normalized $\mathbf{b}^3$ tally is half this.

As the Table 3.4 summation property extends to define the BC vector for n candidates, the normalized $\mathbf{b}^n$ outcome is the sums of pairwise outcomes divided by $(n - 1)$. To illustrate, the pairwise tallies of Table 3.2 for $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$ are, respectively 5:5, 5:5, 5:5, 5:5, 4:6, 5:5, so the BC vector tallies for all candidates are $(5+5+5, 5+4+5, 5+5+5, 5+6+5)$ with a $\mathbf{b}^4$ tally $(5, 4\frac{2}{3}, 5, 5\frac{1}{3})$. Similarly, the $\mathbf{b}^3$ tally for $\{A, B, C\}$ is $\frac{1}{2}(5 + 5, 5 + 5, 5 + 5)$ while that for $\{B, C, D\}$ is $\frac{1}{2}(5 + 4, 5 + 5, 6 + 5)$.

The $\mathbf{E}^4_2$ and $\mathbf{E}^4_3$ tallies are, respectively $(0, 7, 1, 2)$ and $(0, 0, 7, 3)$, so according to Eq. 3.3 the $\mathbf{w}^4$ tally is

$$(5, 4\frac{2}{3}, 5, 5\frac{1}{3}) + \alpha_2(0, 7, 1, 2) + \alpha_3(0, 0, 7, 3). \tag{3.5}$$

So, a $\mathbf{w}^4$ with a larger α_2 value assists B, while one with more α_3 emphasis helps C.

3.3 Graphing election outcomes

I now use this Eq. 3.2 profile and Eq. 3.5 to introduce a new, easy way to determine all possible election outcomes resulting from the same profile. Note that the Eq. 3.5 election tallies are expressed in terms of α_2, α_3; i.e., in terms of the positional method. By plotting the α_2, α_3 values causing ties between different pairs, we determine all election outcomes (the 18 different strict outcomes is the maximum number (Saari [11])) generated by this profile. For instance, a $B \sim C$ outcome occurs if and only if a procedure satisfies $4\frac{2}{3} + 7\alpha_2 = 5 + \alpha_2 + 7\alpha_3$, or if $6\alpha_2 - 7\alpha_3 = 1/3$. So, for instance, all $\mathbf{w}^4 = \mathbf{b}^4 + \alpha_2\mathbf{E}^4_2 + \alpha_3\mathbf{E}^4_3$ procedures where $6\alpha_2 - 7\alpha_3 > 1/3$ have the relative $B \succ C$ outcome for this profile.

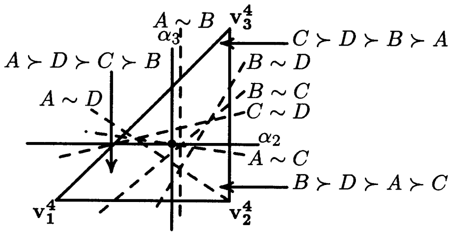

Figure 1. Positional hull

The results of these computations are graphed in Figure 1 where the bulleted center point is the BC. The regions for three strict rankings are indicated; to find the 15 others, reverse a pairwise ranking when crossing the appropriate indifference line. In this way we discover, for instance, that only D avoids being bottom ranked by some procedure. Any four-candidate profile can be similarly analyzed to find all outcomes with all associated positional methods.

3.4 Connections

Of particular importance, relationships similar to that between the BC and pairwise elections hold between the $\mathbf{w}_\lambda^k$ election tallies and certain $\mathscr{P}^s$ entries for $s > k$ (Saari [13]). Namely, a summation argument identical to that used for Table 3.4 shows that when the same $\mathbf{w}^k$ is used with all k-candidate subsets, it defines voting vectors for subsets with more candidates. For instance, the plurality election outcomes over all three-candidate sets uniquely determine the $(3,1,0,0)$ four-candidate outcome, while $(1,1,0)$ determines the $(3,3,2,0)$ election rankings.

To show the $(1,1,0)$ and $(3,3,2,0)$ relationship, compute the number of points a voter with preferences $A \succ B \succ C \succ D$ assigns each candidate with $(1,1,0)$ over the four three-candidate elections. As shown in Table 3.6, this is $(3,3,2,0)$.

	A	B	C	D
$\{A,B,C\}$	1	1	0	
$\{A,B,D\}$	1	1		0
$\{A,C,D\}$	1		1	0
$\{B,C,D\}$		1	1	0
Total	3	3	2	0

(3.6)

This summation approach, then, associates the three-candidate plurality vector $\mathbf{v}_1^3$ with the four-candidate voting vector $\frac{1}{3}(3,1,0,0) = (1,\frac{1}{3},0,0)$ and $\mathbf{v}_2^3 = (1,1,0)$ with $(1,1,\frac{2}{3},0)$. So, because the Eq. 3.2 plurality tallies for $\{A,B,C\}$, $\{A,B,D\}$, $\{A,C,D\}$, $\{B,C,D\}$ are, respectively, $(5,3,2)$, $(5,2,3)$, $(5,2,3)$, $(2,3,5)$, the $(3,1,0,0)$ tally of $(5+5+5, 3+2+2, 2+2+3, 3+3+5)$ is found by adding a candidate's tallies over her three three-candidate plurality elections.

More generally, the summation approach defines a mapping

$$g_k : \mathscr{P}^{k-1} \to \mathscr{P}^k, \quad k = 3, \ldots, n. \tag{3.7}$$

Here $g_k(\mathbf{w}^{k-1})$ is the k-candidate voting vector for set S defined by the number of points a voter assigns each candidate in S by using $\mathbf{w}^{k-1}$ over all $(k-1)$-candidate subsets of S. Because g_k is a linear mapping, we have for $\mathbf{w}^{k-1} = \sum_{j=1}^{k-2} \lambda_j \mathbf{v}_j^{k-1}$ that

$$g_k(\mathbf{w}^{k-1}) = g_k\left(\sum_{j=1}^{k-2} \lambda_j \mathbf{v}_j^{k-1}\right) = \sum_{j=1}^{k-1} \lambda_j g_k(\mathbf{v}_j^{k-1}).$$

In turn, this identifies the class of k-candidate voting vectors that can be expressed as

$$\mathbf{w}^k = \sum_{j=1}^{k-2} \lambda_j g_k(\mathbf{v}_j^{k-1}), \quad \sum_{j=1}^{k-2} \lambda_j = 1. \tag{3.8}$$

The only restriction imposed on the scalars $\{\lambda_j\}_{j=1}^{k-2}$ in Eq. 3.8 is that they define a voting vector. (Negative λ_j values are admissible.) In turn, the election outcomes of the Eq. 3.8 voting vectors are uniquely determined by the $\{\lambda_j\}_{j=1}^{k-2}$ values and the $\{\mathbf{v}_j^{k-1}\}_{j=1}^{k-2}$ election tallies. These dependencies define valuable ordering and consistency properties.

Definition 1. *The* derived set *of voting vectors in* $\mathscr{P}^k$*, denoted by* $\mathscr{D}^k$*, consists of all voting vectors that can be expressed in the form of Eq. 3.8*

The following theorem collects properties of the derived set $\mathscr{D}^k$ and the pyramid $\mathscr{P}^k$.

Theorem 3. *For* $k \geq 3$*, the* voting pyramid $\mathscr{P}^k$ *is the* $(k-2)$*-dimensional convex hull of the* k*-candidate voting vectors* $\{\mathbf{v}_j^k\}_{j=1}^{k-1}$*. The BC voting vector,* $\mathbf{b}^k$*, is at the barycenter of* $\mathscr{P}^k$*.*

The derived set $\mathscr{D}^k$ *is spanned by the vectors*

$$\mathbf{u}_j^k = g_k(\mathbf{v}_j^{k-1}) = \frac{1}{k-1}(k-1, \ldots, k-1, j, 0, \ldots, 0), \quad j = 1, \ldots, k-2, \tag{3.9}$$

where the j *value is in the* $(j+1)$ *coordinate position.* $\mathscr{D}^k$ *is a* $(k-3)$*-dimensional subspace which includes the BC voting vector* $\mathbf{b}^k$*. A normal vector for the* $\mathscr{D}^k$ *affine space in* $\mathscr{P}^k$*, called the* departure vector, *is*

$$\mathbf{d}^k = \left(0, \binom{k-1}{1}, -\binom{k-1}{2}, \ldots, (-1)^{k-1}\binom{k-1}{k-2}, 0\right) \tag{3.10}$$

Figure 2 displays the derived set within $\mathscr{P}^n$. The departure vector, which is orthogonal to the derived set, points into the portion of the pyramid opposite that of the plurality vector. In the $n = 4$ portion of Figure 2, it points from the bulleted barycenter toward $\mathbf{v}_2^4$; in the $n = 5$ portion it points to the side with $\mathbf{v}_2^5, \mathbf{v}_4^5$.

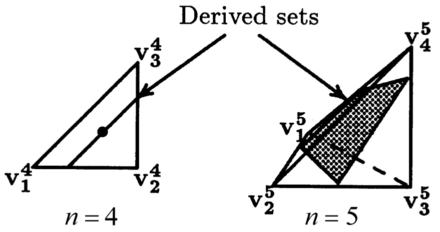

Figure 2. Voting pyramids $\mathscr{P}^n$

3.5 Election relationships

A standard difficulty in analyzing profiles is the need to individually compute the outcome for each procedure. With Theorem 3, no longer is this necessary.

Corollary 1. *For a given n-candidate profile* $\mathbf{p}$*, all possible positional election rankings for all subsets of candidates are uniquely determined by the pairwise election outcomes and the departure vector* $\mathbf{d}^{|S_j|}$ *outcomes for each* S_j*.*

The induction proof is immediate. The pairwise outcomes define all $\mathbf{b}^{|S_j|}$ outcomes. As all three-candidate voting vectors can be expressed as $\mathbf{w}^3 = \mathbf{b}^3 + \mu\mathbf{d}^3$, the election outcome for a set is the $\mathbf{b}^3$ tally plus μ times the $\mathbf{d}^3$ tally. This determines all $\{\mathbf{v}_j^3\}_{j=1}^2$ tallies, which, in turn, determines the $\mathbf{u}_j^4$ outcomes. These $\mathbf{u}_j^4$ tallies, along with the departure vector $\mathbf{d}^4$ outcomes, determine all four-candidate tallies. By induction, all possible outcomes are found.

To illustrate Corollary 1 with the Table 3.2 profile, we already have shown how the pairwise elections determine all $\mathbf{b}^n$ tallies. The $\mathbf{d}^3 = (0, 2, 0)$ tallies of a three candidate subset is twice the number of times each candidate is in *second place* (in the three-candidate rankings), while the $\mathbf{d}^4 = (0, 3, -3, 0)$ outcome is three times the difference between how often a candidate is in second and third place. So, the departure vector $\mathbf{d}^n$ tallies of $\{A, B, C\}$, $\{A, B, C, D\}$ for Eq. 3.2 are, respectively, $2(0, 4, 6)$ and $3(0 - 0, 7 - 0, 1 - 7, 2 - 3)$.

Because (Theorem 3) $\mathscr{D}^3 = \{\mathbf{b}^3\}$, Corollary 1 ensures that all positional outcomes are weighted sums of the $\mathbf{d}^3$ and $\mathbf{b}^3$ tallies. For instance, because

$$\mathbf{b}^3 - \frac{1}{4}\mathbf{d}^3 = \left(1, \frac{1}{2}, 0\right) - \frac{1}{4}(0, 2, 0) = (1, 0, 0), \tag{3.11}$$

the plurality tally of $\{A, B, C\}$ for Eq. 3.2 is $(5, 5, 5) - \frac{1}{4}(0, 8, 12) = (5, 3, 2)$. More generally, $\mathbf{w}_s^3 = (1, s, 0) = \mathbf{b}^3 + \frac{1}{2}(s - \frac{1}{2})\mathbf{d}^3$, so the $\mathbf{w}_s^3$ outcome is $(5, 3 + 4s, 2 + 6s)$. By varying $s \in [0, 1]$, we obtain the *procedure line* of election outcomes.

A similar computation identifies the four-candidate outcomes. Because

$$\mathbf{v}_2^4 = (1, 1, 0, 0) = \mathbf{b}^4 + \frac{1}{9}\mathbf{d}^4 = \left(1, \frac{2}{3}, \frac{1}{3}, 0\right) + \frac{1}{9}(0, 3, -3, 0),$$

the $\mathbf{v}_2^4$ outcome for Table 3.2 is $(5, 4\frac{2}{3}, 5, 5\frac{1}{3}) + \frac{1}{9}(0, 21, -18, -3) = (5, 7, 3, 5)$. All remaining four-candidate outcomes come from the computation of the $\mathbf{u}_j^4$

vertices as determined by the three-candidate tallies. In Figure 1, the outcomes for procedures in the derived set are on the line through the origin (the BC outcome) orthogonal to the vector pointing to the $\mathbf{v}_2^4$ vertex (the vertex in the lower right corner). All election outcomes on this line are strictly determined by appropriate three-candidate tallies; differences in the four-candidate tallies and outcomes differ are due to the departure vector $\mathbf{d}^4$.

An immediate Corollary 1 consequence is that only the BC outcomes are related to the pairwise tallies. The tallies of all other procedures are distanced from the pairwise outcomes through the departure $\mathbf{d}^k$ tallies. This structure provides a new, conceptually simple explanation for the result that *the BC outcomes must be related to the pairwise ranking, but the rankings of any other procedure need not be related in any manner!* (The first part is due to Nanson [6]; the second part was found, with very different techniques, by Saari [12]. Sieberg [17] also noted and used this separation effect with a statistical interpretation.)

Theorem 3 helps identify α^n system voting vectors (see Theorem 2) which enjoy the following election relationships. In this statement, let $g_{k+s}(\mathbf{w}^k)$ be the voting vector for subsets with $(k+s)$ candidates defined by the s-step iterative process $g_{k+s}(g_{k+s-1}(\ldots(g_{k+1}(\mathbf{w}^k))\ldots))$.

Corollary 2. *Let* $\mathbf{w}^k$ *be a voting vector,* $2 \leq k < n$, *and let* s *satisfy* $1 \leq s \leq n-k$. *When all* k*-candidate elections are tallied with* $\mathbf{w}^k$ *and all* $(k+s)$*-candidate elections with* $g_{k+s}(\mathbf{w}^k)$, *the election outcomes satisfy the following relationships.*

1. *A candidate who is top-ranked in all* $\mathbf{w}^k$ *elections cannot be bottom-ranked in the* $(k+s)$*-candidate election tallied with* $g_{k+s}(\mathbf{w}^k)$.
2. *A candidate who is bottom-ranked in all* $\mathbf{w}^k$ *elections cannot be top-ranked in the* $g_{k+s}(\mathbf{w}^k)$ *election. This candidate is* $g_{k+s}(\mathbf{w}^k)$ *strictly ranked below a candidate who always is* $\mathbf{w}^k$ *top-ranked.*
3. *If all* $\mathbf{w}^k$ *outcomes end in a complete tie, than the* $g_{k+s}(\mathbf{w}^k)$ *outcome also is a complete tie.*

To illustrate, if a candidate wins all three-candidate plurality elections, she cannot be bottom ranked in the $g_4(\mathbf{v}_1^3) = \mathbf{u}_1^4 = \frac{1}{3}(3,1,0,0)$ election. If all three-candidate antiplurality elections are tied (using $\mathbf{v}_2^3 = (1,1,0)$), the four-candidate election tallied with $g_4(\mathbf{v}_2^3) = \mathbf{u}_2^4 = \frac{1}{3}(3,3,2,0)$ also is tied. (The plurality or $\mathbf{u}_1^4$ outcomes need not be tied.) These assertions hold for a $(k+s)$-candidate set S and its k-candidate subsets.

Proof. The proof of this important result, which significantly extends similar BC assertions, is trivial. A candidate who always is $\mathbf{w}^k$ top-ranked receives more than the average number of total votes cast over all k-candidate elections; consequently, she cannot be $g_{k+s}(\mathbf{w}^k)$ bottom-ranked. The proof of the second assertion is similar. The third assertion requires the same number of points to be added for each candidate. □

A natural way to avoid inconsistencies and election paradoxes, then, is to avoid procedures which involve departure vector, $\mathbf{d}^k$, effects. To do so, tally all

k-candidate elections with $\mathbf{w}^k$, the $(k+1)$-candidate elections with $g_{k+1}(\mathbf{w}^k)$, the $(k+2)$-candidate elections with $g_{k+2}(g_{k+1}(\mathbf{w}^k))$, $\ldots$. It follows that the fewest paradoxes along with the ultimate consistency and the largest number of election relationships require starting with the smallest value of $k = 2$. This is the Borda Count; this explains why the BC admits more consistency in election outcomes and more election relationships than any other positional procedure.

3.6 Paradoxes and examples

Theorem 3 and its corollaries provide unlimited opportunities to generate new paradoxes while explaining why they occur. For instance, a procedure which rewards a voter's top-ranked candidate but penalizes his bottom-ranked candidate is the five-candidate voting vector $\mathbf{v}_1^5 + \mathbf{v}_4^5 = (2, 1, 1, 1, 0)$. Its normalized form $\mathbf{w}^5 = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = \frac{2}{3}\mathbf{u}_1^5 - \frac{1}{3}\mathbf{u}_2^5 + \frac{2}{3}\mathbf{u}_3^5$ ensures that $\mathbf{w}^5 \in \mathscr{D}^5$ and (from Eq. 3.9) that its outcomes are uniquely determined by the three scalars and the election tallies of the four-candidate procedures $\mathbf{v}_1^4$, $\mathbf{v}_2^4$, $\mathbf{v}_3^4$.

The inclusion $\mathbf{w}^5 \in \mathscr{D}^5$ guarantees election relationships, but the negative scalar $(-\frac{1}{3})$ suggests that not all relationships are "positive." After all, the negative $\mathbf{u}_2^5$ coefficient requires the $\mathbf{v}_2^4 = (1, 1, 0, 0)$ tallies to be *subtracted* from the sum of the two other four-candidate elections. Causing further doubt about $\mathbf{w}^5$ is that Eqs. 3.8, 3.7 allow a candidate's $\mathbf{w}^5$ tally to be interpreted as coming from the sums of her four-candidate tallies with the procedure $\frac{2}{3}\mathbf{v}_1^4 - \frac{1}{3}\mathbf{v}_2^4 + \frac{2}{3}\mathbf{v}_3^4 = (1, \frac{1}{3}, \frac{2}{3}, 0)$. But, $(1, \frac{1}{3}, \frac{2}{3}, 0)$ is *not* a voting vector because this perverse method assigns twice as many points to a third-place candidate as to a second ranked one. Thus one of the promised $(2, 1, 1, 1, , 0)$ relationships is to *penalize* a candidate who often is second-ranked in four-candidate subsets over a candidate who is consistently third-ranked. Consequently, a candidate who always is the $\mathbf{v}_2^4 = (1, 1, 0, 0)$ winner over the four-candidate subsets by virtue of being the voters' second choice could do poorly in the five-candidate $(2, 1, 1, 1, 0)$ election outcome.

3.6.1 Helping Condorcet losers

Note that $g_5((1, \frac{1}{2}, \frac{1}{2}, 0)) = g_5(\frac{1}{2}(1, 0, 0, 0) + \frac{1}{2}(1, 1, 1, 0)) = \frac{1}{2}(\mathbf{u}_1^5 + \mathbf{u}_3^5)$ rewards a candidate if a voter has her top-ranked in *four-candidate* elections and penalizes her if she is bottom-ranked. To introduce perversity, modify this procedure by *penalizing* a candidate who does well in pairwise elections. As $\mathbf{b}^5$ is the sum of pairwise votes, one choice is $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = \mathbf{u}_1^5 + \mathbf{u}_3^5 - \mathbf{b}^5 = \mathbf{w}^5$. So another disturbing $\mathbf{w}^5 \in \alpha^5$ election relationship is that $\mathbf{w}^5$ *punishes* candidates who do well in pairwise elections. Namely, $\mathbf{w}^5$ rewards the Condorcet loser while hurting the Condorcet winner. The derivation of this surprising assertion is not restricted to $\mathbf{w}^5$; once the $\mathscr{D}^k$ dimension at least unity (so $k \geq 4$), *any* $\mathscr{D}^k$ voting vector other than $\mathbf{b}^k$ can be expressed as a sum where the Borda point is subtracted. This allows an argument resembling the one for $\mathbf{w}^5$ to be fashioned for *any* non-BC

$\mathscr{D}^k$ voting vector; i.e., the election relationships promised by Theorem 3 can be negative.

Corollary 3. *For a non-BC voting vector* $\mathbf{w}^k \in \mathscr{D}^k$, $k \geq 4$, *there exists a* $(k-1)$ *candidate voting procedure* $\mathbf{w}^{k-1}$ *so that* $\mathbf{w}^k$ *rewards positive* $\mathbf{w}^{k-1}$ *outcomes at the expense of penalizing a strong pairwise performance.*

Proof. For such a $\mathbf{w}^k \in \mathscr{D}^k$, there is a $\mathbf{w}^{k-1}$ so that $g_k(\mathbf{w}^{k-1})$ is in the interior of the line segment with vertices $\mathbf{w}^k$ and $\mathbf{b}^k$. Thus there is a $\mu \in (0,1)$ so that $\mu\mathbf{b}^k + (1-\mu)\mathbf{w}^k = g_k(\mathbf{w}^{k-1})$. The conclusion follows by solving for $\mathbf{w}^k$. □

3.6.2 A useful result

Even more disturbing outcomes are generated by the voting vectors *off* of the derived set $\mathscr{D}^n$. Here the departure vector $\mathbf{d}^n$ can modify the outcomes in any manner without regard for the election outcomes of subsets. So, while the outcomes of a non-BC procedure in $\mathscr{D}^n$ may *subtract* each candidate's pairwise tallies from the tallies of other procedures, the $\mathbf{d}^k$ component ignores – hence can differ significantly from – the outcomes of other procedures. In other words, procedures which involve a departure term suffer all faults of α^n procedures as well as encountering new difficulties.

A vector relationship between the plurality $\mathbf{v}_1^k$ and $\mathbf{d}^k$ leads to Corollary 4, an important tool introduced as (an unproved) Proposition 1 in [15]. This result, which asserts that all election pathologies are reflected by the commonly used plurality vote, plays a central role in the current discussion. Let $Plur(\mathbf{p}, |S_j|)$ be the vector plurality tally of subset S_j for profile $\mathbf{p}$.

Corollary 4. *For* $n \geq 3$ *candidates and a specified* $\mathbf{w}^{S_k}$, *there are constants* $a^{|S_j|}$ *so that the* $\mathbf{w}^{S_k}$ *tally of profile* $\mathbf{p}$ *is*

$$\sum_{S_j \subset S_k} a^{|S_j|} Plur(\mathbf{p}, |S_j|). \tag{3.12}$$

Proof. The plurality vector (and other $\{\mathbf{v}_j^k\}_{j=1}^{k-1}$ vectors) is not in $\mathscr{D}^k$. (This follows from the form of the $\mathbf{u}_j^k$ vectors.) The derived set $\mathscr{D}^k$ is a dimension lower than $\mathscr{P}^k$, so normal vector $\mathbf{d}^k$ is a linear combination of $\mathscr{D}^k$ vectors and the plurality $\mathbf{v}_1^k$. (See, for instance, Eq. 3.11.) In turn, a profile's $\mathbf{d}^k$ outcome, needed for Corollary 1, uses the same linear combination of the profile's plurality and specified $\mathbf{u}_j^k$ tallies. The $\mathbf{u}_j^k$ outcomes are determined by the outcomes of $\mathbf{d}^{k-1}$ and $\mathscr{D}^{k-1}$ entries from $(k-1)$-candidate subsets of S_s. But each $\mathbf{d}^{k-1}$ outcome is determined from the $(k-1)$ plurality and $\mathscr{D}^{k-1}$ tallies of the specified subsets. Tracing the obvious induction argument to its base, we discover that *all* election tallies can be obtained from the pairwise and plurality tallies of subsets of S_s. □

3.7 Positive behavior

Corollary 4, along with Theorem 1, motivates the following assertion.

Corollary 5. *For $n \geq 3$, a necessary condition for a system vector $\mathbf{W}^n$ to be in α^n is that at least one of the voting vectors in $\mathbf{W}^n$ is in a derived set. A sufficient condition is if $\mathbf{w}^k$ is used with all k-candidate subsets of S and $g_{k+j}(\mathbf{w}^k)$ is used with the $(k+j)$-candidate subset S for some $j \geq 1$.*

So, election relationships occur only with the summation process motivated by Eq. 3.4. This is necessary, not sufficient; e.g., no relationships among the $\mathbf{W}^4$ rankings occur if the four-candidate election is tallied with $(3, 1, 0, 0) \in \mathscr{D}^4$ while each of the four triplets is tallied with one of $(1, 0, 0)$, $(1, \frac{1}{4}, 0), (1, \frac{2}{3}, 0), (1, 1, 0)$. On the other hand, should three of the triplets be tallied with the same voting vector, then restrictions are imposed upon the rankings of the sole candidate in these three subsets. If this common voting vector is on the plurality side of the BC, and if this candidate is top-ranked in all two and three candidate elections, she cannot be bottom ranked in the full four-candidate election.

As a partial summary, we have established and explained the following:

1. The Borda Count offers more consistency of election outcomes over all subsets of candidates than any other positional procedure.
2. Election relationships among rankings require using voting vectors from the various $\mathscr{D}^k$ sets.

Namely, *only the BC is spared a negative Corollary 3 conclusion.*

4 Profiles

As demonstrated, much more goes wrong with voting outcomes than previously suspected. All of this perverse behavior is explained with the profile decomposition. This decomposition uses *profile differentials* introduced in (Saari [15]; they are the differences between profiles with the same number of voters. While this requires *negative* numbers of voters to have certain preferences, it creates no problems in computing election outcomes.

The profile differentials are divided into classes determined by how they effect different procedures. I occasionally use the *space of normalized profiles* where, instead of specifying the number of voters of a particular type, the fraction of all voters is used. This space is identified with the $n! - 1$ dimensional simplex

$$Si(n!) = \left\{ \mathbf{x} = (x_1, \ldots, x_{n!}) \in R^{n!} \mid \sum_{j=1}^{n!} x_j = 1,\ x_j \geq 0 \right\} \tag{4.1}$$

4.1 Universal Kernel

The companion paper [15] identifies the *Kernel* profile differentials which have no effect on any pairwise or positional election outcome for any subset of candidates. In other words, they behave like the profile $\mathbf{K}^n$ which has one voter for each of the $n!$ types. A surprising fact is that as the number of candidates increases, the dimension of the subspace of Kernel profiles rapidly dominates profile space. Already for $n = 5$, the Kernel dimension is over half that of profile space, and when $n = 7$, it consumes over 93% of the dimensions. The following theorem was stated and used in [15]; it is proved here.

Theorem 4. *(Saari [15]) For $n \geq 3$, there exists a $n! - 2^{n-1}(n-2) - 2$ dimensional subspace $\mathcal{UK}^n$ of the profile space $Si(n!)$, called* the universal kernel, *where if $\mathbf{p} \in \mathcal{UK}^n$, then its word for all $\mathbf{W}^n$ choices is a complete tie for each subset of candidates.*

A characterization of the Kernel profile differentials for $n = 3, 4$ and a description of them for $n \geq 5$ is in [15].

4.2 Basic profiles

The Basic profiles and their fundamental importance are reviewed next. For details, see [15].

Definition 2. *A n-candidate* Basic profile differential for candidate c_j, *denoted by $\mathbf{B}^n_j$, has a voter for each ranking where c_j is top-ranked and -1 voters for each ranking where c_j is bottom-ranked.*

The following [15] result identifies the source of the remarkable Basic profile properties.

Theorem 5. *For each subset of candidates, the $\mathbf{B}^n_j$ tallies of all normalized positional voting procedures agree. In particular, for $k \geq 2$, if c_j is in a k-candidate subset of candidates, she receives $(n-1)!$ points and each of the other candidates receives $-\frac{(n-1)!}{k-1}$ points. If c_j is not in a k-candidate subset, then all candidates receive zero points.*

The Basic profile differentials define a $(n-1)$-dimensional subspace spanned by any $(n-1)$ of $\{\mathbf{B}^n_{c_j}\}^n_{j=1}$. These differentials satisfy the equality $\sum_{j=1}^n \mathbf{B}^n_{c_j} = \mathbf{0}$.

To describe this result in another manner, let $\tau^{\mathbf{w}^{|S|}}(c_i, c_j)$ be the difference between c_i's and c_j's $\mathbf{w}^{|S|}$ election tallies for profile $\mathbf{p}$. If S is the pair $\{c_i, c_j\}$, then use $\tau^2(c_i, c_j)$. Also, let $\mu_k = (n-1)![1 + ((k-1)!)^{-1}]$. For Basic profile for any subset S that contains c_i and c_j, and for any $\mathbf{w}^{|S|}$, it follows from Theorem 5 that

$$\tau^{\mathbf{w}^{|S|}}(c_i, c_j) = (a_i^B - a_j^B)\mu_{|S|}. \tag{4.2}$$

Because Eq. 4.2 holds for any subset of candidates, the c_i, c_j relative ranking *remains the same for all subsets of candidates (including pairs) and for all choices of tallying procedures.* So, Basic profiles avoid the earlier described problems with non-transitive pairwise rankings, where election outcomes for set S can change with the choice of $\mathbf{w}^{|S|}$ and where the rankings change when candidates are added or dropped.

Going beyond ensuring transitive pairwise outcomes, the Basic profiles satisfy the following stronger *additive transitivity* condition.

Definition 3. *(Saari [15]) A profile* **p** *satisfies* additive transitivity *if for any subset $S = \{c_1, c_2, \dots, c_k\}$ of candidates tallied with any $\mathbf{w}^S$ and for any permutation of the indices, we have that*

$$\tau^{\mathbf{w}^S}(c_1, c_2) + \cdots + \tau^{\mathbf{w}^S}(c_{j-1}, c_j) = \tau^{\mathbf{w}^S}(c_1, c_{j+1}). \tag{4.3}$$

As a special case, for any collection of candidates $\{c_j, c_k, c_s\}$, we have

$$\tau^2(c_j, c_k) + \tau^2(c_k, c_s) = \tau^2(c_j, c_s). \tag{4.4}$$

To prove a Basic profile satisfies Eq. 4.4, notice from Eq. 4.2 that the left side equals $(a_j - a_k)\mu_2 + (a_k - a_s)\mu_2$ which equals $(a_j - a_s)\mu_2 = \tau^2(c_j, c_s)$. The proof of Eq. 4.3 is the same.

This additive transitivity condition is satisfied even when candidates $\{c_1, c_{k+1}\}$ are linked by a “chain” in the sense that for any collection of subsets $S_1, \dots, S_{k+1}$, if $c_1, c_2 \in S_1, c_2, c_3 \in S_2, \dots, c_k, c_{k+1} \in S_k$, then $c_1, c_{k+1} \in S_{k+1}$. For any $\mathbf{w}^{|S_j|}$ assigned to S_j, $j = 1, \dots, k+1$, the result is that $\tau^{\mathbf{w}^{|S_{k+1}|}}(c_1, c_{k+1})$ is a normalized sum of the differences in tallies of the chain of pairs connecting c_1 with c_{k+1}; this is true for all choices of subsets and positional methods. In particular,

$$a_1^B - a_{k+1}^B = \frac{\tau^{\mathbf{w}^{|S_{k+1}|}}(c_1, c_{k+1})}{\mu_{|S_{k+1}|}} = \sum_{j=1}^{k} \frac{\tau^{\mathbf{w}^{|S_j|}}(c_j, c_{j+1})}{\mu_{|S_j|}} = \sum_{j=1}^{k}(a_j^B - a_{j+1}^B). \tag{4.5}$$

This amazing property is false in general; it fails even for the unanimity profile.

4.3 Condorcet profiles

The *Condorcet profile differentials* are defined in terms of *Condorcet n-tuples*. To construct and define a Concordet n-tuple, attach a *ranking disk* (a disk that rotates about its center) to a fixed background. Equally spaced along the disk’s circular boundary place the ranking numbers $1, 2, \dots, n$. To represent a ranking r of the candidates, place each candidate’s name on the fixed background next to the appropriate ranking number. Rotate the disk in a clockwise direction until the number 1 points to the next candidate; the new location of the numbers define a second ranking. Continue this process until n rankings are defined. The resulting set of n rankings (this is a Z_n orbit) defines the *Condorcet n-tuple defined by ranking r* which is denoted by $\mathbf{R}_r^n$. Let $\rho(r)$ be the ranking which completely reverses r; e.g., $\rho(A \succ C \succ D \succ B) = B \succ D \succ C \succ A$.

Definition 4. *The Condorcet profile differential defined by ranking* r, $\mathbf{C}^n_r$, *is where one voter is assigned to each* $\mathbf{R}^n_r$ *ranking and* -1 *voters are assigned to each* $\mathbf{R}^n_{\rho(r)}$ *ranking.*

For two candidates c_i *and* c_j, *the* $c_i \succ c_j$ Condorcet profile differential *is the sum of all* $\mathbf{C}^n_r$ *differentials defined by an* r *where* c_i *is top-ranked and* c_j *is second-ranked.*

As developed in (Saari [15]), the Condorcet profile differentials are completely responsible for all problems with pairwise rankings and tallies, and for all election oddities of procedures using this information. They even cause trouble with positional rankings of subsets of candidates. The reason is that with Condorcet profile differentials, the pairwise vote discards valuable information about the rationality of the voters. In particular, the pairwise vote cannot distinguish whether the data comes from the $\mathbf{C}^n_r$ transitive preferences, or a large number of indistinguishable (for the pairwise vote) profiles involving voters with cyclic preferences where the cycles are "natural" outcomes. The cycles, then, reflect the attempt of the pairwise vote to meet the needs of non-existent irrational (i.e., a voter without transitive preferences) voters rather than the actual rational ones. A similar argument explains many of the positional voting problems.

Most properties of Condorcet profiles are developed in [15]. New results needed here follow.

Theorem 6. *Assume there are* $n \geq 4$ *candidates.*

1. *For a* k*-candidate subset, the departure vector* $\mathbf{d}^k$ *tally of* $\mathbf{C}^n_r$ *is zero if* $k = n$ *or if* k *is an odd integer. If* $k < n$ *is an even integer, the* $\mathbf{d}^k$ *tally need not be zero.*
2. *Each* $c_i \succ c_j$ *Condorcet profile differential is orthogonal to the Basic vectors as well as to all* $\mathbf{d}^k$ *vectors for each subset of three or more candidates. Thus, the set of Basic and* $c_i \succ c_j$ *Condorcet vectors uniquely determine all pairwise and all BC outcomes. For each subset of* k *candidates, a positional outcome based on these profiles agrees with the BC outcome.*
3. *For the* $c_i \succ c_j$ *Condorcet profile differential,* c_i *beats* c_j *in a pairwise election with the* $(n-2)(n-2)! : -(n-2)(n-2)!$ *tally. However,* c_j *beats, and* c_i *loses, to all other candidates with a* $(n-2)! : -(n-2)!$ *tally. The pairwise outcome for any other pair of candidates is a tie where each candidate receives zero votes.*
4. *For a* k*-candidate subset,* $2 < k < n$, *the tally of a* $c_i \succ c_j$ *Condorcet profile differential is the same for all positional methods. If both* c_i *and* c_j *are in the set, then* c_i *receives* $(n-k)(n-2)!$ *points,* c_j *receives the negative of this, and all other candidates receive zero points. If* c_i *is in the set, but* c_j *is not, then* c_i *receives* $-(k-1)(n-2)!$ *points while each other candidate receives* $(n-2)!$ *points. If* c_j *is in the set when* c_i *is not, then* c_j *receives* $(k-1)(n-2)!$ *points and each other candidate receives* $-(n-2)!$ *points. For all other sets, all candidates receive zero points.*

So, the pairwise votes are determined by the Basic (or Borda, see Saari [15]) and Condorcet profile differentials. The Basic profile retains the rationality of voters for all procedures; the Condorcet portion explicitly drops the individual rationality assumption for the pairwise vote. The stronger the Condorcet portion (relative to the Basic part) of a profile, the more the pairwise outcomes reflects a loss of the assumption of individual transitivity. Part 1 shows that the Condorcet term can alter certain positional outcomes (i.e., procedures defined with a departure vector). Important for our analysis is the Part 2 assertion that this difficulty is avoided with the $c_i \succ c_j$ Condorcet differentials because they have a zero tally with departure vectors. This prevents these Condorcet terms from further affecting positional outcomes; indeed, (part 4) the tallies of all positional methods remain the same. Part 3 shows that the $c_i \succ c_j$ Condorcet terms divide into cyclic triplets. In what follows, the "$c_i \succ c_j$ Condorcet space" is the space spanned by all $c_i \succ c_j$ Condorcet profile differentials; $1 \leq i < j \leq n$.

5 All candidates

Starting with Borda's seminal paper [2], many choice issues compare positional rankings of all n candidates with the associated pairwise outcomes. For the most part, results are severely limited by the complexity of the analysis. With an appropriate profile decomposition, however, the analysis becomes fairly easy and straightforward. To introduce the approach and the appropriate profile differentials, I first identify a problem that re-occurs throughout this analysis.

5.1 Natural coordinates

According to Proposition 1, all positional outcomes can be described in terms of the pairwise (which determine the BC outcome) and $\{\mathbf{E}^n_k\}_{k=2}^{n-1}$ tallies. This suggests using the following differentials.

Definition 5. *Let $\mathscr{E}^n_{k,j}$ be the profile differential with $(n-1)$ voters assigned to each ranking where c_j is kth ranked and -1 voters to each remaining ranking.*

Because $\mathbf{E}^n_k$ only counts how many times a candidate is ranked in kth position, $\mathscr{E}^n_{k,j}$ emphasizes c_j by listing her, and only her, in this position. Part 1 of Theorem 7 shows that the ($(n-1)$-dimensional) space defined by $\{\mathscr{E}^n_{k,j}\}_{j=1}^n$ completely characterizes $\mathbf{E}^n_k$ outcomes because an orthogonal profile has no effect on $\mathbf{E}^n_k$ outcomes.

Theorem 7. *Assume $n \geq 3$.*

1. *If $\mathbf{p}$ is a profile differential orthogonal to the $(n-1)$-dimensional subspace spanned by $\{\mathscr{E}^n_{k,j}\}_{j=1}^n$, then the $\mathbf{E}^n_k$ tally of $\mathbf{p}$ is a complete tie where each candidate receives zero votes.*
2. *The space defined in Part 1 is spanned by any $(n-1)$ of the $\mathscr{E}^n_{k,j}$ choices; these differentials satisfy $\sum_{j=1}^n \mathscr{E}^n_{k,j} = \mathbf{0}$.*

3. *The* $\mathbf{E}^n_k$ *tally of* $\mathscr{E}^n_{k,j}$ *assigns* $(n-1)(n-1)!$ *points to* c_j *and* $-(n-1)!$ *points to each remaining candidate.*
4. *For* $i \neq k$, *the space spanned by* $\{\mathscr{E}^n_{k,j}\}^n_{j=1}$ *is* not *orthogonal to the space spanned by* $\{\mathscr{E}^n_{i,j}\}^n_{j=1}$, *nor to the space of Basic profiles. Indeed, while the pairwise tally of* $\mathscr{E}^n_{k,j}$ *has a zero-zero tie for all pairs not involving* c_j, *in a* $\{c_j, c_s\}$ *election,* c_j *receives* $n(n-2)!(n+1-2k)/2$ *votes and* c_s *receives the negative of this. In a* $\mathbf{E}^n_i$ *election,* $i \neq k$, *the outcome has* c_j *bottom ranked (with a* $-(n-1)!$ *tally) and everyone else tied for top-ranked where each candidate has a* $(n-2)!$ *tally.*

According to part 4, the $\mathscr{E}^n_{k,j}$ profiles are flawed because they influence elections other than those for the kth ranked candidate. We could adjust for this flaw if the subsets were ordered in a manner where a profile for each set affects the election outcomes only of the set and of higher ordered sets, but not any other set of candidates. (When this is true, to achieve a desired election outcome, first select profiles for the lower ordered sets. Then, adjust the profiles assigned to the higher ordered sets to compensate for the effects of already selected profiles. This is illustrated with an example after Corollary 7.) As the $\mathscr{E}^n_{k,j}$ profiles do not permit such an ordering, I develop an alternative profile system which impacts on desired procedures (i.e., each $\mathbf{E}^n_k$) but not on others (e.g., the other $\mathbf{E}^n_i$, $i \neq k$ and the pairwise vote). This lack of orthogonality among differentials is standard; e.g., it arises in (Saari [15]) where the natural Borda profiles affect pairwise *and* positional election outcomes. The Basic profiles, and their strong, delightful properties, result from choosing an appropriate orthogonal basis.

5.2 Orthogonal basis

I first identify profile differentials which affect a specified class of positional procedures *but not pairwise and other election outcomes.* This construction shows that, with the sole exception of the BC, positional and pairwise rankings need not be related in any manner.

Definition 6. *For* $n \geq 3$ *candidates and* $2 \leq k < \frac{n+1}{2}$, *the* kth place Symmetric profile differential *for* c_j, $\mathbf{S}^n_{k,j}$, *is where one voter is assigned to each ranking where* c_j *is* k*th ranked and to the reversal of this ranking;* -1 *voters are assigned to each ranking where* c_j *is top-ranked and to the reversal of this ranking. For* $k = (n+1)/2$, *two voters are assigned to each ranking where* c_j *is* k*th ranked, and* -1 *voters to each ranking where* c_j *is either top- or bottom-ranked.*

For $2 \leq k \leq \frac{n-1}{2}$, *the* kth place Alternating profile differential *for* c_j, $\mathbf{A}^n_{k,j}$, *is where one voter is assigned to each ranking where* c_j *is* k*th ranked,* -1 *voters for the reversal of this ranking,* $-(n+1-2k)/(n-1)$ *voters are assigned to each ranking where* c_j *is top-ranked, and* $(n+1-2k)/(n-1)$ *voters for the reversal of this ranking. For* $k = (n+1)/2$, $\mathbf{A}^n_{k,j} = \mathbf{0}$.

Fractions are avoided by using $(n-1)\mathbf{A}^n_{k,j}$ and $2\mathbf{S}^n_{(n+1)/2,j}$. The next result establishes the Alternating and Symmetric differentials as appropriate building blocks for differentials. Indeed, they played a role in the discovery of the departure profiles in Section 6.

Theorem 8. *Assume there are* $n \geq 3$ *candidates.*

1. *The pairwise tally of any pair* $\{c_i, c_s\}$ *with an Alternating or Symmetric profile is zero.*
2. *For* $k \neq (n+1)/2$, *the* $(\mathbf{E}^n_k + \mathbf{E}^n_{n+1-k})$ *tally of* $\mathbf{S}^n_{k,j}$ *gives* $2(n-1)!$ *votes to* c_j *and* $-2(n-2)!$ *votes to each remaining candidate. The* $(\mathbf{E}^n_k + \mathbf{E}^n_{n-(k-1)})$ *tally of any other Symmetric or Alternating profile differential assigns zero points to each candidate. For* $k = (n+1)/2$, *the same assertions hold for* $\mathbf{E}^n_k$.
3. *The* $(\mathbf{E}^n_k - \mathbf{E}^n_{n-(k-1)})$ *tally of* $\mathbf{A}^n_{k,j}$ *assigns* $2(n-1)!$ *points to* c_j *and* $-2(n-2)!$ *votes to each of the remaining candidates. The* $(\mathbf{E}^n_k - \mathbf{E}^n_{n-(k-1)})$ *tally of any other Symmetric or Alternating profile differential is a zero tie.*
4. *For fixed n and k, we have that* $\sum_{j=1}^n \mathbf{S}^n_{k,j} = \sum_{j=1}^n \mathbf{A}^n_{k,j} = \mathbf{0}$.

Proposition 1 suggests the definition of *n-candidate Positional differentials* affected only by $\mathbf{E}^n_j$.

$$\mathbf{P}^n_{k,j} = \begin{cases} \mathbf{S}^n_{k,j} + \mathbf{A}^n_{k,j} & \text{for } 2 \leq k < (n+1)/2; \\ \mathbf{S}^n_{k,j} - \mathbf{A}^n_{k,j} & \text{for } (n+1)/2 < k \leq n = 1, \\ \mathbf{S}^n_k & \text{for } k = (n+1)/2. \end{cases} \tag{5.1}$$

A direct computation proves the following assertion.

Proposition 2. *For fixed* n, k, *and* j, $\mathbf{P}^n_{k,j}$ *assigns*

- *2 voters for each ranking where* c_j *is kth ranked,*
- $-2(n-k)/(n-1)$ *voters for each ranking where* c_j *is top-ranked, and*
- $2(1-k)/(n-1)$ *voters for each ranking where* c_j *is bottom-ranked.*

For $n = 3$, these differentials are the Reversal profiles of (Saari [10, 16]). The following statement is central for the rest of this section.

Corollary 6. *With* $\mathbf{P}^n_{k,j}$, *the pairwise tally of each pair of candidates is a zero-zero tie.*

The $\mathbf{E}^n_k$ *tally of* $\mathbf{P}^n_{k,j}$ *assigns* $2(n-1)!$ *points to* c_j *and* $-2(n-2)!$ *points to each remaining candidate. The* $\mathbf{E}^n_k$ *tally of* $\mathbf{P}^n_{i,j}$, $i \neq k$, *assigns zero points to each candidate.*

For each k, the n-candidate Positional differentials, $\{\mathbf{P}^n_{k,j}\}^n_{j=1}$, *define a* $(n-1)$ *dimensional space where any* $(n-1)$ *of the differentials serve as a basis. For fixed* n, k, $\sum_{j=1}^n \mathbf{P}^n_{k,j} = \mathbf{0}$.

For any choice of coefficients $\{\beta_j\}^n_{j=1}$, $\mathbf{E}^n_k$ *has identical tallies for the two profiles*

$$\frac{2}{n-1} \sum_{j=1}^n \beta_j \mathscr{E}^n_{k,j}, \quad \sum_{j=1}^n \beta_j \mathbf{P}^n_{k,j}. \tag{5.2}$$

This corollary asserts that the Positional differentials have the desired property of influencing the outcome for only the specified candidate with the specified $\mathbf{E}_k^n$. Moreover, because Eq. 5.2 identifies profiles defined in terms of $\mathscr{E}_{k,j}^n$ with those in terms of Positional differentials, it follows that any $\mathbf{E}_k^n$ outcome obtained with one system of differentials is obtained with the other system. From the perspective of understanding election outcomes, then, it does not matter which system is used. But with a choice, the fact the Positional differentials $\{\mathbf{P}_{k,j}^n\}$ do not affect other $\mathbf{E}_s^n$ or pairwise tallies makes them much easier to use. A minor cost is that the Positional differentials do not satisfy Part 1 of Theorem 7. (This is expected; to define an orthogonal system, profiles affecting positional outcomes must be subtracted.)

Proof. The proof of all but the last two assertions follows from Theorem 8 and the relationships $\mathbf{E}_k^n = \frac{1}{2}((\mathbf{E}_k^n + \mathbf{E}_{n+1-k}^n)) + (\mathbf{E}_k^n - \mathbf{E}_{n+1-k}^n); \mathbf{E}_{n+1-k}^n = \frac{1}{2}((\mathbf{E}_k^n + \mathbf{E}_{n+1-k}^n) - (\mathbf{E}_k^n - \mathbf{E}_{n+1-k}^n))$.

That the Positional differentials span a space of at least dimension $(n-1)$ follows from the form of the differentials. The next to last assertion follows from Part 4 of Theorem 8. The last assertion follows from the linearity of the tallying process and by using the obvious scaling required by Part 3 of Theorem 7. □

5.3 New results

A sample of new results or proofs which follow from Corollary 6 follows.

Corollary 7. *(Saari [12]) For $n \geq 3$ candidates, let $\mathbf{w}^n \neq \mathbf{b}^n$. Choose any ranking for the n candidates and any ranking for each of the $\binom{n}{2}$ pairs of candidates. There exists a profile so that the pairwise and the $\mathbf{w}^n$ rankings are the selected ones.*

When this result from (Saari [12]) was discovered, it was not known how to find supporting profiles. The following simpler proof outlines this construction.

Proof. It is shown in Saari [15] how to construct a profile $\mathbf{p}_1$ so that ranking for each pair is as selected. With Corollary 6, we can construct a profile $\mathbf{p}_2$ which has no effect on pairwise rankings but which changes the $\mathbf{w}^n$ ranking to the desired one. Profile $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ satisfies the assertion. □

To illustrate Corollary 7, suppose the Basic and Condorcet profiles used to create the desired pairwise rankings define the $\mathbf{b}^6$ tally $(10, 5, 6, 3, 1, 0)$. Although the Basic profiles influence both pairwise and the positional outcomes (by determining the BC outcome), the Positional profiles have no effect upon the pairs so they can be chosen to alter the positional outcomes in any desired manner. Namely, to alter this profile so that the $\mathbf{w}^6 = (1, 1, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, 0) = \mathbf{b}^6 + \frac{1}{3}\mathbf{E}_2^6 - \frac{1}{3}\mathbf{E}_3^6$ outcome is the almost reversed $c_6 \succ c_5 \succ c_4 \succ c_3 \succ c_2 \succ c_1$, select $\beta_{k,j}$ coefficients for

$$\sum_{j=1}^{6} \beta_{2,j}\mathbf{P}^6_{2,j} + \beta_{3,j}\mathbf{P}^6_{3,j}$$

to have the desired ranking from the tally

$$\begin{aligned} 10 \quad &+ \quad \frac{2}{3}[5!(\beta_{2,1} - \beta 3,1) - 4! \left[\sum_{j\neq 1}(\beta_{2,j} - \beta_{3,j})\right], \ldots, 0 \\ &+ \quad \frac{2}{3}\left[5!(\beta_{2,6} - \beta_{3,6}) - 4! \left(\sum_{j\neq 6}(\beta_{2,j} - \beta_{3,j})\right]\right). \end{aligned}$$

This is easy; e.g., according to Corollary 6, the $\mathbf{p} = \sum_{j=1}^{6} 2j\mathbf{P}^6_{2,j}$ coefficients force the $\mathbf{E}^6_2$ outcome to be the desired ranking. As this tally overwhelms the BC tally, adding $\mathbf{p}$ to the original profile creates a desired example without using $\mathbf{P}^6_{3,j}$ terms.

For a more striking illustration, I use Corollary 6 to prove there exists (by constructing) a four-candidate profile where *each* candidate is in top, second, third, and bottom place with appropriate positional methods. Divide the candidates into two pairs and assign each pair to a $\mathbf{E}^4_k$, $k = 2, 3$; e.g., assign $\{A, D\}$ to $\mathbf{E}^4_2$ and $\{B, C\}$ to $\mathbf{E}^4_3$. As the $\mathbf{E}^4_k$ rankings reflect the ordering of the $\sum \beta_{k,j}\mathbf{P}^4_{k,j}$ coefficients, Positional differentials can be designed so one candidate from a pair wins and the other loses a $\mathbf{E}^4_k$ election. But changing the α_k sign reverses the $\mathbf{E}^4_k$ ranking (Eq. 3.3) so that the previous loser now wins; this ensures the conclusion. To be specific, let $\mathbf{p}_2 = 2\mathbf{P}^4_{2,A} + \mathbf{P}^4_{2,B} + 0\mathbf{P}^4_{2,C} - \mathbf{P}^4_{2,D}$ and $\mathbf{p}_3 = 3\mathbf{P}_{3,B} + \mathbf{P}_{3,A} - \mathbf{P}^4_{3,D} - 2\mathbf{P}^4_{3,C}$. The following uses Corollary 6 to simplify the computation of $\mathbf{p} = \mathbf{p}_2 + \mathbf{p}_3$ tallies.

$\mathbf{E}^4_2$ tally	A	B	C	D	$\mathbf{E}^4_3$ tally	A	B	C	D
$2\mathbf{P}^4_{2,A}$	24	−8	−8	−8	$3\mathbf{P}_{3,B}$	−12	36	−12	−12
$\mathbf{P}^4_{2,B}$	−4	12	−4	−4	$\mathbf{P}_{3,A}$	12	−4	−4	−4
$-\mathbf{P}^4_{2,D}$	4	4	4	−12	$-\mathbf{P}_{3,D}$	4	4	4	−12
					$-2\mathbf{P}_{3,C}$	8	8	−24	8
Total	24	8	−8	−24		12	44	−36	−20

(5.3)

For $\mathbf{p} = \mathbf{p}_2 + \mathbf{p}_3$, the $\mathbf{w}^4_{\alpha_2,\alpha_3} = \mathbf{b}^4 + \alpha_2\mathbf{E}^4_2 + \alpha_3\mathbf{E}^4_3$ coefficients require, for $\alpha_3 = 0$ and $\alpha_2 > 0$, the outcome $A \succ B \succ C \succ D$ (dictated by the ordering of the $\mathbf{p}_2$ coefficients), and the reverse $D \succ C \succ B \succ A$ if $\alpha_2 < 0$. Similarly, if $\alpha_2 = 0$, then $\alpha_3 > 0$ gives the $B \succ A \succ D \succ C$ ranking (from the ordering of the $\mathbf{p}_3$ coefficients) while $\alpha_3 < 0$ procedures reverse this ranking. So, $\mathbf{p}$ satisfies the conditions. All of $\mathbf{p}$'s positional rankings (12 are strict) are in Figure 3a (with a Figure 1 analysis) while Figure 3b displays $3\mathbf{p}_2$ as computed from Proposition 2.

More generally, we obtain the following new assertion proving that pairwise and positional rankings can be highly uncoordinated.

Theorem 9. *For $n > 3$ alternatives, there exists a profile* $\mathbf{p}$ *where, with appropriate positional methods, each candidate c_j is in top, second, …, last position; $j = 1, \ldots, n$.*

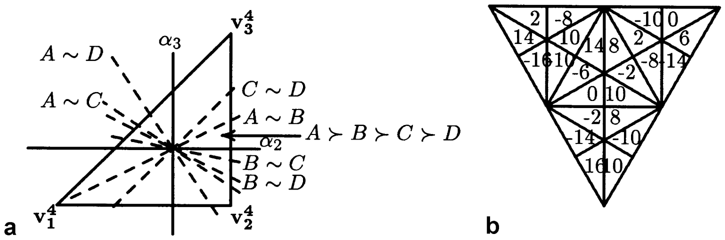

Figure 3a,b. Constructing examples. **a** Positional rankings; **b** $3\mathbf{p}_2 = \mathbf{3}[\mathbf{2P}^4_{2,A} + \mathbf{P}^4_{2,B} - \mathbf{P}^4_{2,D}]$

Choose any ranking for each of the pairs of candidates. There exists a profile where the above conclusion holds and each pairwise ranking is the selected one.

As suggested by the $n = 4$ example, the proof (Section 9) assigns a certain ranking to each $\mathbf{E}^n_k$; this defines two rankings for positional procedures depending on the sign of the $\mathbf{E}^n_k$ coefficient in Eq. 3.3. If n is even or odd, this assignment uses, respectively, only $n/2$ or $(n + 1)/2$ of the procedures. Consequently, for $n \geq 6$, extra $\mathbf{E}^n_k$ rankings remain free to be chosen so that the above can be accompanied with even more surprising outcomes.

5.4 More positional rankings

Slight modifications of the above create a surprising number of new positional rankings. To explain, the Figure 1 analysis finds the plane of procedures with a pairwise tie for each pair of candidates. As Positional differentials do not affect pairwise votes, they do not affect the BC outcome. Therefore (according to Eq. 3.3), all planes pass through the BC tally of a complete tie as illustrated in Figure 3a.

When the BC outcome is not a complete tie, the planes of procedures no long pass through the BC (as the BC outcome is not tied), so the planes move. Each plane keeps the same slope (determined by the Positional differential tallies), but it is translated in a direction determined by which candidate from the pair has the advantage in the $\mathbf{b}^n$ tally. This translation forces the planes to cross one another in a complicated manner, and the intersections create new regions. Each new region defines a new positional ranking admitted admitted by the profile.

To see this with $n = 3$, the line in Figure 4a represents the tallies of a Positional differential. As required, this line passes through the BC outcome of complete indifference. According to Eq. 3.3, by adding Basic terms (affecting pairwise and Borda tallies), the $\mathbf{b}^3$ outcome translates the procedure line of Figure 4a. In the unlikely (lower dimensional) case that this BC outcome is along the original line, the translated procedure line allows (at most) three different positional rankings. If the $\mathbf{b}^3$ tally has any other value, the procedure line is translated into one of the positions indicated in Figure 4b. In particular, if the Positional tallies sufficiently overwhelm those from the BC, the new procedure line passes through seven different ranking regions. One choice (the upper dashed

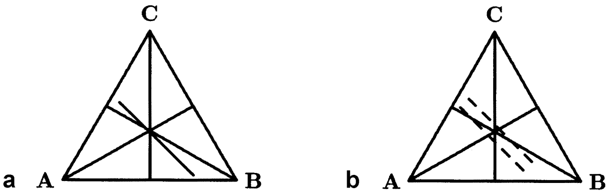

Figure 4a,b. Adding BC to positional profiles

line in Figure 4b) requires each candidate to "lose" with some procedure; only A cannot be a winner. The lower dashed line is where each candidate "wins" with an appropriate procedure. (See the "procedure line" in (Saari [10].)

To understand what happens with $n = 4$ candidates, start with the special case of a BC $A \succ [B \sim C \sim D]$ outcome. When computing the line of procedures causing a $A \sim B$ outcome, A's larger tally (coming from the BC) moves the original $A \sim B$ line to the left. By slightly moving this line in Figure 3a, its new location must cross the $B \sim C, B \sim D, C \sim D$ lines to generate at least two new strict positional rankings. The $A \sim C, A \sim D$ lines also are moved (both to the left) to create a proliferation of new positional outcomes. More general outcomes occur if a BC outcome has no ties. This analysis applied to n candidates recaptures the results of (Saari [11]) but with stronger conclusions and a much simpler analysis.

5.5 Maximizing the number of positional outcomes

In this manner, profiles can be constructed which admit the maximum number of election rankings with changes in the positional procedure. To review, start with a Positional profile where the plane of procedures giving a tie vote for each pair meet only at the BC outcome. Choose this Positional profile so that, by changing the $\mathbf{w}^n$, each candidate is ranked in each position. Break this degenerate situation by adding a voter whose strict preferences disagree with the rankings defined by the Positional procedures. As illustrated in Figure 4 for the special case of three candidates, this change moves the procedure hull off of the complete indifference point so it intersects a maximum number of ranking regions.

5.6 Associated results

The power of this approach is that the Condorcet portion of the profile does not affect positional tallies, and the $\mathbf{P}^n_{k,j}$ differentials have no effect on the pairwise vote. Consequently, the pairwise and positional outcomes can be designed independent of one another. The following provides a flavor of the many new results that now are immediate. The first comment significantly extends a well known result of Condorcet [3] showing that the Condorcet winner need not be top-ranked by any positional procedure.

Theorem 10. *For $n \geq 3$ alternatives, there exist profiles where a Condorcet winner exists, yet she is ranked either at the bottom, or next to the bottom for all positional methods. Indeed, for each j satisfying $1 \leq j \leq (n-1)$, there are profiles with a Condorcet winner where all positional methods have the Condorcet winner jth ranked. For each j, $1 \leq j \leq (n-1)$, a profile can be found where for each $i, j \leq i \leq n$, the Condorcet winner is ith ranked with some positional procedures, but the Condorcet winner never is ranked higher than jth.*

For 10 candidates, choose any ranking for each pair. There exists a profile so that the pairwise outcome is the selected ranking and there are more than 84 million different rankings as the choice of the positional method changes.

Proof. Once the BC outcome is determined, $\mathbf{P}^n_{k,j}$ differentials can be selected to achieve any desired outcome. The BC outcome is determined by the Basic profile. The Condorcet profile can alter the pairwise outcomes to achieve any admissible ranking. If the rankings define the Condorcet winner, than this winner cannot be BC ranked lower than next to the bottom. The first conclusion now follows.

For the second conclusion, select Basic and Condorcet profiles to attain the desired pairwise rankings. Next, follow the above construction to ensure the positional outcome conclusion. □

6 The departure profiles

While useful, the $\{\mathbf{P}^n_{k,j}\}$ differentials are inappropriate for describing positional outcomes over *all* subsets of candidates. This is because the derived sets (Section 3) require the $\mathbf{w}^s$ outcomes for s candidates to determine the $g_{s+1}(\mathbf{w}^s), g_{s+2}(g_{s+1}(\mathbf{w}^s)), \ldots$ outcomes for larger sets of candidates. According to the voting pyramid structure (Theorem 3), the new profiles determine departure vector $\mathbf{d}^k$ outcomes without affecting the outcomes of procedures in the derived set. This allows outcomes for positional methods with $\mathbf{d}^k$ components to deviate as wildly as desired from conclusions of the BC, pairwise, and methods in the derived set.

These *departure profiles* (defined next) affect the rankings of subsets with more candidates. But, these subsets are ordered by set inclusion, so the profiles can be used as given. Namely, as illustrated with interaction of Positional and Borda outcomes, first determine the outcome for a subset of candidates, and adjust the outcome for a superset. An alternative approach is to use standard procedures to design an orthogonal basis to remove extraneous influences. These procedures are well understood, so I emphasize the original profile differentials where the rankings of the subsets are adjusted. I illustrate both approaches with $n = 4$ candidates.

Definition 7. *The c_i* departure profile differential for the set of all $n \geq 3$ candidates, $\mathbf{D}^n_{c_i}$ *is defined by assigning $(-1)^s \binom{n-1}{s-1}$ voters to each ranking where c_j is sth ranked, $s = 1, \ldots, n$.*

For a k-candidate subset $\mathscr{S}$, $3 \le k < n$, the c_i departure profile differential $\mathbf{D}^k_{c_i,\mathscr{S}}$ *is where the $\mathbf{D}^k_{c_i}$ profile is augmented $(n-k)!$ times by adding all possible rankings of the candidates not in $\mathscr{S}$ ranked below those in S.*

The $\mathbf{D}^n_{c_i}$ form is sufficiently bizarre to suggests that it is highly unlikely to ever arise. But, remember, to be affected by $\mathbf{D}^n_{c_i}$, a profile only needs to have $\mathbf{D}^n_{c_i}$ components. To prove this is a common occurrence, notice in Figure 5 that both $\mathbf{D}^3_{A,\mathscr{S}}$ for $\mathscr{S} = \{A, B, C\}$ and $\mathbf{D}^4_A$ have entries in the $A \succ B \succ C \succ D$ ranking region. Consequently, the profile decomposition of the *unanimity profile* $A \succ B \succ C \succ D$ has components in the $\mathbf{D}^3_{A,\{A,B,C\}}$ and $\mathbf{D}^4_A$ directions; i.e., even the unanimity profile experiences affects of the departure directions. Important properties of these departure profiles differentials are introduced in the next theorem.

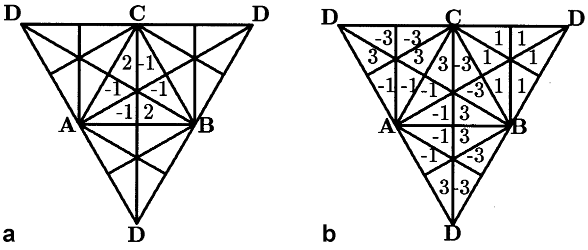

Figure 5a,b. Departure profiles. **a** $\mathbf{D}^3_{\mathbf{A},\mathbf{S}}$. **b** $\mathbf{D}^4_{\mathbf{A}}$

Theorem 11. *For $n \ge 3$ candidates, let $\mathscr{S}$ be a subset of k candidates, $3 \le k \le n$. The following statements concern the tally of $\mathbf{D}^k_{c_i,\mathscr{S}}$, $c_i \in \mathscr{S}$.*

1. *All pairwise tallies end in a complete tie where each candidate receives zero points.*
2. *If at least one candidate in $\mathscr{S}$ is not in $\mathscr{S}'$, then all $\mathscr{S}'$ positional tallies of $\mathbf{D}^k_{c_i,\mathscr{S}}$ are zero.*
3. *If $\mathbf{w}^k \in \mathscr{D}^k$ for set $\mathscr{S}$, then the $\mathbf{w}^k$ tally of $\mathbf{D}^k_{c_i,\mathscr{S}}$ is a complete tie where all candidates receive zero points.*
4. *Let $\mathscr{S}$ is a proper subset of the set of candidates $\mathscr{S}'$ where the departure vector for $\mathscr{S}'$ is $\mathbf{d}'$. The $\mathbf{d}'$ tally of $\mathbf{D}^k_{c_i,\mathscr{S}}$ has a positive tally for c_i, and an equal negative tally for all remaining candidates in $\mathscr{S}$. If $\mathscr{S}'$ has one more candidate than $\mathscr{S}$, then this candidate has a zero tally. In general, each candidate in $\mathscr{S}' - \mathscr{S}$ has the same tally. The sum of the tallies is zero*
5. *The $\mathscr{S}$ plurality tally for c_i has her bottom ranked with $-(n-k)!(k-1)!$ votes. The tally for each of the remaining $(k-1)$ candidates is $(n-k)!(k-2)!$ votes.*
6. *If $\mathbf{d}^k$ is the departure vector for $\mathscr{S}$, then the $\mathbf{d}^k$ tally of $\mathbf{D}^k_{c_i,\mathscr{S}}$ has c_i top ranked with tally $\gamma_{n,k} = (n-k)! \sum_{s=2}^{k-1} [\binom{k-1}{s-1}]^2$ and all remaining candidates tied for bottom. As the sum of the votes equals zero, a bottom ranked candidate's tally is the $\frac{-1}{k-1}$ multiple of c_i's tally.*

7. *The departure profile differentials* $\{\mathbf{D}^k_{c_i,\mathcal{S}}\}_{c_i\in\mathcal{S}}$ *span a space of dimension* $|\mathcal{S}|-1$ *determined by any* $(|\mathcal{S}|-1)$ *of them. They satisfy the relationship*

$$\sum_{c_j\in\mathcal{S}} \mathbf{D}^k_{c_j,\mathcal{S}} = \mathbf{0}. \tag{6.1}$$

According to Parts 1 and 3, the departure differentials have no affect upon pairwise or positional outcomes from the derived set; they only influence the departure direction. Parts 2 and 4 prove that the departure profiles affect outcomes only in supersets of candidates. Corollary 4 identifies the plurality vote as an important procedure defined by the departure vector, so Part 5 specifies the plurality tally of a departure differential. The departure tally of a departure profile is in Part 6, while Part 7 identifies the subspace structure of the departure profiles within profile space.

To indicate why a natural election outcome for $\mathbf{D}^k_{c_i,\mathcal{S}}$ is a completely tied vote, start with a two person profile with preferences r and $\rho(r)$. As in Section 5, the opposite preferences suggests a cancellation of votes causing a complete tie. But, for an odd integer n, $\mathbf{D}^n_{c_i}$ is the sum of two-person profiles of the $(r,\rho(r))$ type. Indeed, if r is a ranking where c_i is jth ranked, then $\rho(r)$ is a ranking where c_i is $(n-j)$th ranked. This one-to-one relationship and the equal number of voters for each setting completes the proof of this assertion. The argument for an even number of voters involves aspects of the alternating differentials of Section 5.

A related argument that $\mathbf{D}^k_{c_i,\mathcal{S}}$ should lead to a complete tie comes from parts 1, 2 of Theorem 11; they assert that $\mathbf{D}^k_{c_i,\mathcal{S}}$ has no influence on the pairs, on the rankings of any subset, or on the rankings of any other subset with the same number of candidates. For instance, with ten candidates, $\mathbf{D}^{10}_{c_i}$ has a completely tied tie vote for all sets of pairs, triplets, . . . , sets of nine candidates, and then, in stark contradiction, it suddenly ranks c_i as the top-ranked candidate for the set of all ten-candidates. Justifying such a conclusion is not easy.

7 Applications

To indicate implications, I use Theorem 11 in three ways. The first illustrates how to create profiles with certain desired behavior. The second indicates how to analyze voting procedures. The third underscores the complexity of standard profiles.

7.1 Constructing profiles

This paper starts with Theorems 1, 2 which generalize all assertions and examples describing how election outcomes change when candidates are added or dropped. I now provide a sharper statement where the proof outlines the construction of all supporting profiles. This assertion is a direct consequence of Theorem 11.

Theorem 12. *Suppose each voting vector in the system vector* $\mathbf{W}^n$ *has a component in the departure direction. Then,* $\mathbf{W}^n \notin \alpha^n$. *Indeed, for each subset of candidates, select a ranking in any desired manner. There exists a profile* $\mathbf{p}$ *where, for each subset, the specified ranking is the election ranking. Namely, the word* $\tilde{F}(\mathbf{p}, \mathbf{W}^n)$ *consists of specified rankings.*

Proof. The induction proof is simple. Consider subsets of $k = 2$ candidates. The specified pairwise rankings are determined by the Basic and $c_i \succ c_j$ Condorcet differentials. These rankings determine the BC outcomes for each subset.

For specified k, $2 < k < n$, assume a profile is constructed where the indicated election outcome occurs for the specified positional procedure for all subsets of j candidates, $2 \leq j \leq k$. In turn, the election outcomes of procedures in the derived sets of each $(k + 1)$-candidate subset are determined. For each $(k + 1)$-candidate subset, the specified voting vector has a component in the departure direction. Consequently, the departure profiles for each of these sets can be selected so that the indicated election outcome occurs for the specified positional method. Because these positional procedures have no affect on the election outcomes for other subsets of $(k + 1)$ candidates and for any subset with fewer than $(k+1)$ candidates, the new profile creates the desired election outcomes for the specified positional method for all subsets up to $(k + 1)$ candidates. This completes the induction proof. □

To be concrete, suppose we want to create a four-candidate profile with a complete tie for all pairwise, all BC, and all four-candidate positional rankings. The plurality rankings of the triplets, however, are to form a cycle where, say, $A \succ B \sim C$, $B \succ C \sim D$, $C \succ D \sim A$, $D \succ A \sim B$. According to Theorem 11, part 5, the construction starts with

$$\mathbf{p}' = -\mathbf{D}^3_{A,\{A,B,C\}} - \mathbf{D}^3_{B,\{B,C,D\}} - \mathbf{D}^3_{C,\{C,D,A\}} - \mathbf{D}^3_{D,\{D,A,B\}} \tag{7.1}$$

which is illustrated in Figure 6 a.

According to parts 1, 2 of Theorem 11, $\mathbf{p}'$ has no impact on pairwise or BC votes; all end in complete ties with zero tallies. Part 2 asserts that the $\mathbf{D}^3_{A,\mathscr{S}}$ portion for each triplet $\mathscr{S}$ has no effect on the outcome for any other triplet. Thus, the plurality triplet outcomes are as desired. While Part 4 warns that $\mathbf{p}'$ might change the positional outcomes of the set of four candidates, the $\mathbf{p}'$ symmetry creates a cancellation leaving the four-candidate plurality ranking in a complete tie. All of this can be verified from Figure 6 with the tallying procedures introduced in [15]. (In Section 8, the $\mathbf{p}'$ properties are generalized.)

(To convert $\mathbf{p}'$ into a profile, add an appropriate $\mathscr{UK}^4$ profile to make all terms non-negative. For instance, add $2\mathbf{K}^4$ and then divide all entries by 3 to generate a 12 voter profile with the desired properties.)

To indicate how to use the ordering of subsets by set inclusion to handle settings where a four-candidate cancellation does not occur, change the $\{A, B, C\}$ ranking of the Figure 6a example to $C \succ A \succ B$; e.g., replace $-\mathbf{D}^3_{A,\{A,B,C\}}$ in $\mathbf{p}'$ with $\frac{1}{3}[-\mathbf{D}^3_{C,\{A,B,C\}} + \mathbf{D}^3_{A,\{A,B,C\}}]$. The new profile is in Figure 6b where the

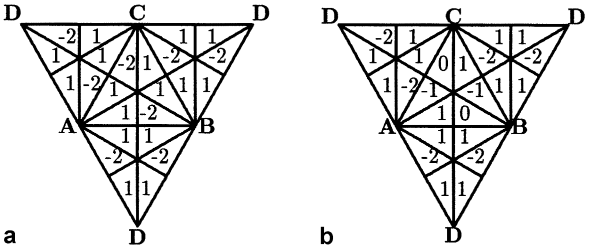

Figure 6a,b. Profiles examples. **a** Example profile. **b** $C \succ A \succ B$ ranking

differences are in the $\{A, B, C\}$ face. This choice, however, changes the four candidate plurality ranking from a complete tie to $C \succ A \sim B \succ D$ with tally 2:0:0:-2. To obtain the desired four-candidate complete tie, add $\frac{1}{4}[\mathbf{D}^4_C - \mathbf{D}^4_A]$. This profile differential does not effect pairwise or triplet rankings (Theorem 11), but it returns the four-candidate ranking to a complete tie.

An alternative approach is to replace the $\mathbf{D}^3_{c_j,\mathscr{S}}$ profiles with a profile that only influences $\mathscr{S}$ rankings. To do this, notice from Figure 5a that the $\mathbf{D}^3_{A,\{A,B,C\}}$ profile has the four candidate plurality ranking of $B \sim C \succ D \succ A$ with the tally $1 : 1 : 0 : -2$. To remove this influence while retaining the $\mathbf{D}^3_{A,\{A,B,C\}}$ properties, add $\frac{1}{4}[-\mathbf{D}^4_A + \frac{1}{2}[\mathbf{D}^4_B + \mathbf{D}^4_C]$ to cancel the four-candidate effects. To remove fractions, use $\tilde{\mathbf{D}}^3_{A,\{A,B,C\}} = 4\mathbf{D}^3_{A,\{A,B,C\}} - \mathbf{D}^4_A + \frac{1}{2}[\mathbf{D}^4_B + \mathbf{D}^4_C]$. This $\tilde{\mathbf{D}}^3_{A,\{A,B,C\}}$ differential, given in Figure 7, has the desired properties of Theorem 11 *plus* the added property that it does not change the four candidate outcomes (as shown by computing the tallies). The choices for other sets and candidates follows from symmetry.

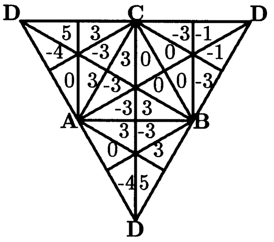

Figure 7. Profiles $\tilde{\mathbf{D}}^3_{A,ABC} = 4\mathbf{D}^3_{\mathbf{A},\mathbf{ABC}} - \mathbf{D}^4_{\mathbf{A}} + \frac{1}{2}[\mathbf{D}^4_{\mathbf{B}} + \mathbf{D}^4_{\mathbf{C}}]$

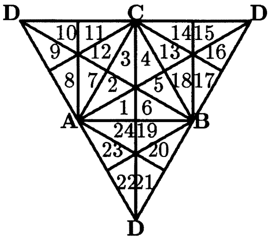

Figure 8. Voter type labels for 4-candidates

7.2 Other procedures

The use of Theorem 11 to analyze other procedures mimics how the Basic and Condorcet profiles are used to analyze procedures based on binary rankings ([15]). Namely, the Basic profile requires agreement over all subsets of candidates; all reasonable procedures behave as we might hope. Conflict is generated only by adding profile deviations. As we now know how to alter the positional rankings of any subset of candidates, we know how to create examples illustrating all flaws of all procedures.

7.2.1 Runoffs

These comments can be illustrated with a five-candidate plurality runoff, where, at each stage, only the bottom ranked candidate is dropped. Start with a Basic profile with the universal ranking $A \succ B \succ C \succ D \succ E$. In direct conflict with this ranking, suppose we want E to be the runoff winner and the order of eliminating candidates is A, B, C, D. Suppose (to cast doubt on the runoff procedure) the rankings of all subsets not being voted upon agree with the Basic ranking.

Choose a Basic profile where all binaries have a stronger victory margin than in $\{D, E\}$ election, An appropriate multiple of the $E \succ D$ Condorcet profile differential changes only the $\{D, E\}$ outcome to have $E \succ D$. Add appropriate $\mathbf{D}^3_{C,\{C,D,E\}}$ and $\mathbf{D}^3_{D,\{C,D,E\}}$ multiples to ensure the $E \succ D \succ C$ plurality outcome, while all other triplets retain the Basic ranking.

The rest of the induction is clear. First, add appropriate four candidate departure profile differentials to return the four candidate outcomes to the Basic profile rankings. Then add appropriate profile differentials to convert the $\{B, C, D, E\}$ outcome into $E \succ D \succ C \succ B$. The same is done for the five-candidate election.

A closely related construction, but using only Condorcet profile differentials can identify flaws of the Nanson method. This is a runoff where at each stage the BC bottom ranked candidate is dropped. While the outcome is the Condorcet winner, when one exists, it now is easy to create examples with explanations suggesting this is the wrong candidate. To illustrate, the development of Chapter 5 from [10] proves that Nanson's method is not monotonic. If, however, only Basic profiles are used, the Nanson's method is monotonic. This lack of monotonicity is caused by the Condorcet portion of a profile. In turn, this raises doubts about Nanson's approach.

7.2.2 Condorcet principle

The Condorcet principle judges procedures by whether the Condorcet winner, when one exists, is top-ranked. There is a large literature (see Kelly [5]) describing those procedures which do, or do not satisfy this condition. I offer a simple approach, based on the profile decomposition, which resolves many of these questions. First, we need a condition to capture the sense of a "directional derivative."

Definition 8. *Let $f(\mathbf{p})$ be a procedure which assigns a ranking for profile $\mathbf{p}$. Let $\mathscr{D}_1 \neq \mathscr{D}_2$ be two non-empty subsets of profiles. Procedure f is said to be susceptible to $\mathscr{D}_2$ relative to $\mathscr{D}_1$ if for any $\mathbf{p}_1 \in \mathscr{D}_1$, there exists $\mathbf{p}_2 \in \mathscr{D}_2$ so that $f(\mathbf{p}_1)$ and $f(\mathbf{p}_1 + \mathbf{p}_2)$ have different rankings. We say that f is strongly susceptible to $\mathscr{D}_2$*

if $\mathbf{p}_2$ *can be selected so that if* $f(\mathbf{p}_2)$ *has a top-ranked candidate, then that is the top-ranked candidate in* $f(\mathbf{p}_1 + \mathbf{p}_2)$.

The profile decomposition identifies the procedures affected by the different profile components, so it is easy to verify whether specified methods satisfy this condition. For instance, as a plurality vector has a component in the departure direction, the plurality procedure is strongly susceptible to departure profiles (relative to Basic profiles). Similarly, a plurality runoff , where there is a runoff for the two top-ranked candidates, is strongly susceptible to departure profiles *and* Condorcet profles (relative to Basic profiles). The BC applied to the set of all n candidates is susceptible only to Basic profiles. For subsets of candidates, the BC is susceptible to Basic and Condorcet profiles, but not to departure differentials.

The ease in determining which procedures satisfy different types of susceptibility makes the following theorem a useful tool. It includes as special cases all published results (I know about) concerning which procedures do not satisfy the Condorcet principle.

Theorem 13. *If a procedure is strongly susceptible to any profile component relative to the* $\{$*Basic* $\cup$ *Condorcet*$\}$ *profiles, then it does not satisfy the Condorcet principle. If a procedure is not susceptible to the Condorcet profiles relative to the Basic profiles, it does not satisfy the Condorcet principle.*

Proof. The Condorcet winner is determined strictly by the Basic and Condorcet profiles. So, if a procedure is strongly susceptible to profiles from some other profiles subspace, then a candidate other than the Condorcet winner can be top-ranked. Similarly, by adding a Condorcet portion to a Basic portion of a profile, the identity of the Condorcet winner can change. Consequently, if a procedure does not reflect the Condorcet portion of a profile (as true with, say, the BC), then there are profiles where the procedure does not have the Condorcet winner top-ranked. □

7.3 Inverse and unanimity

To conclude, notice that the number of voters needed to support a profile $\mathbf{p}$ is not correlated with its apparent complexity when $\mathbf{p}$ is described in terms of its profile components. To illustrate, the following four-candidate profile differential has components in the Basic, the Condorcet, departure profiles in each of the four triplet directions and even departure profiles for the set of all four candidates.

$$\mathbf{p} = \frac{1}{12}\{[3\mathbf{B}^4_A + 2\mathbf{B}^4_B + \mathbf{B}^4_C] + 3\mathbf{C}^4_{A \succ B \succ C \succ D} + 4[\mathbf{D}^3_{B,\{A,B,C\}} + \mathbf{D}^3_{C,\{B,C,D\}} + \mathbf{D}^3_{C,\{C,D,A\}} + \mathbf{D}^3_{B,\{A,B,D\}}] - [\mathbf{D}^4_B + 2\mathbf{D}^4_C]\} \tag{7.2}$$

While $\mathbf{p}$ appears to promise all sorts of complications, it is merely a two-voter unanimity profile differential for $A \succ B \succ C \succ D$. It remains to add the appropriate kernel term.

This Eq. 7.2 representation proves that even the unanimity profile has interesting properties. The first bracketed term of Eq. 7.2 is the important Basic profile; it specifies that the natural ranking for this profile is $A \succ B \succ C \succ D$. The Condorcet term (the second term) introduces a twist to the pairwise rankings to explain why the pairwise tallies fail to reflect A's preferred status. (Recall, the pairwise tallies for $\{A, B\}$ and $\{A, C\}$ agree, but the tally for the Basic portion gives A a higher tally in the second election.) The next bracket of four terms is the profile portion which changes the three candidate plurality outcomes; they cause the ranking to be $A \succ B \sim C$ rather than the natural $A \succ B \succ C$ for $\{A, B, C\}$. To explain, the BC outcome remains compatible with expectations, but the $\mathbf{D}^3_{B,\{A,B,C\}}$ portion of the unanimity profile reduces B's Basic tally to create the distorted $A \succ B \sim C$ plurality outcome. A similar explanation holds for the four-candidate elections. Here, the plurality vote changes the Basic profile ranking of $A \succ B \succ C \succ D$ to $A \succ B \sim C \sim D$ because of the profile's component of departure profile in the last bracketed term.

From Eq. 7.2, it is clear that a useful tool is an approach which identifies the t profile differential components of a profile $\mathbf{p}$. For $n = 3$ candidates, the approach is in (Saari [16]). For $n = 4$, I first label the different voter types according to the numbering of Figure 8; e.g., $A \succ B \succ C \succ D$ is a type-one ranking, while $B \succ C \succ D \succ A$ is type-18.

With this notation, a profile $\mathbf{p}$ can be expressed as

$$\mathbf{p} = \mathscr{A}(\mathbf{C})$$

where $\mathscr{A}$ is the 24×24 matrix with columns defined by appropriate profile differentials and where $\mathbf{C}$ is column vector where its 24 entries are the coefficients of the different profile differentials. The components of $\mathbf{p}$ are given by

$$\mathbf{C} = \mathscr{A}^{-1}(\mathbf{p}) \tag{7.3}$$

Theorem 14. *With the ordering of voter types given in Figure 8, the decomposition of a specified profile* $\mathbf{p}$ *is given by Eq. 7.3 where the components of* $\mathbf{C}$ *identify the coefficients for the profile differentials in the following manner:*

- *The first three* $\mathbf{C}$ *components are, respectively, for departure differentials* $\mathbf{D}^4_A, \mathbf{D}^4_B, \mathbf{D}^4_C$. *The coefficient for remaining* $\mathbf{D}^4_D$ *term is determined from Eq. 6.1.*
- *From the fourth to the eleventh, the* $\mathbf{C}$ *components are collected into pairs. These are the coefficients for the departure profiles coming, respectively, from the triples*

  $$\{A, B, C\}, \{A, C, D\}, \{C, B, D\}, \{B, A, D\}.$$

  For each pair assigned to a triplet, the two $\mathbf{C}$ *coefficients are for the first and second candidate in the listed order; e.g., the seventh* $\mathbf{C}$ *component is* $\mathbf{D}^3_{C,\{A,C,D\}}$ *coefficient in the profile decomposition. The remaining coefficients are determined from Eq. 6.1.*
- *The 12, 13, 14 components of* $\mathbf{C}$ *are, respectively, the* $\mathbf{B}^4_A$, $\mathbf{B}^4_B$, $\mathbf{B}^4_C$ *coefficients.*

- *Components 15, 16, 17 are, respectively, for the Condorcet terms*

$$\mathbf{C}^4_{A\succ B\succ C\succ D},\ \mathbf{C}^4_{A\succ B\succ D\succ C},\ \mathbf{C}^4_{A\succ C\succ B\succ D}.$$

- *Component 18 is for* $\mathbf{K}^4$.
- *The final six components are for double reversal kernel vectors (see Saari [15]) given by changes along edges of the representation tetrahedron in the order*

$$\{A,C\},\{C,B\},\{A,B\},\{C,D\},\{B,D\},\{A,D\}.$$

Matrix $\mathscr{A}^{-1}$ *is* $\frac{1}{24}$ *of the following matrix.*

$$\begin{vmatrix}
0 & 0 & -1 & -2 & -2 & -1 & 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & -1 & -2 & -1 & 1 & 2 \\
-1 & -2 & -2 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & -1 & -2 & -1 & 1 & 2 & 2 & 1 & -1 & -2 & -1 & 1 \\
-2 & -1 & 0 & 0 & -1 & -2 & 1 & -1 & -2 & -1 & 1 & 2 & 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & 0 & 0 & 1 & 2 \\
0 & -4 & 4 & 0 & -4 & 4 & -4 & -4 & -4 & 4 & 4 & 4 & 0 & 0 & 0 & -4 & -4 & -4 & 4 & 4 & 4 & 0 & 0 & 0 \\
4 & -4 & 0 & 4 & -4 & 0 & -4 & -4 & -4 & 0 & 0 & 0 & 4 & 4 & 4 & -4 & -4 & -4 & 0 & 0 & 0 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 0 & 0 & -4 & 4 & 0 & -4 & 4 & -4 & -4 & 0 & 0 & 0 & -4 & -4 & 4 & 4 & 4 & -4 & -4 \\
4 & 4 & 0 & 0 & 0 & 4 & 4 & -4 & 0 & 4 & -4 & 0 & -4 & -4 & 4 & 4 & 4 & -4 & -4 & 0 & 0 & 0 & -4 & -4 \\
4 & 0 & 0 & 0 & 4 & 4 & -4 & 4 & 4 & 4 & -4 & -4 & 0 & -4 & 4 & 0 & -4 & 4 & -4 & -4 & 0 & 0 & 0 & -4 \\
0 & 4 & 4 & 4 & 0 & 0 & -4 & 0 & 0 & 0 & -4 & -4 & 4 & -4 & 0 & 4 & -4 & 0 & -4 & -4 & 4 & 4 & 4 & -4 \\
4 & 4 & 4 & 0 & 0 & 0 & -4 & -4 & 0 & 0 & 0 & -4 & -4 & 4 & 4 & 4 & -4 & -4 & 0 & -4 & 4 & 0 & -4 & 4 \\
0 & 0 & 0 & 4 & 4 & 4 & -4 & -4 & 4 & 4 & 4 & -4 & -4 & 0 & 0 & 0 & -4 & -4 & 4 & -4 & 0 & 4 & -4 & 0 \\
3 & 3 & 2 & 1 & 1 & 2 & 2 & 1 & -1 & -2 & -1 & 1 & -1 & -2 & -3 & -3 & -2 & -1 & 1 & -1 & -2 & -1 & 1 & 2 \\
2 & 1 & 1 & 2 & 3 & 3 & -1 & -2 & -3 & -3 & -2 & -1 & 1 & -1 & -2 & -1 & 1 & 2 & 2 & 1 & -1 & -2 & -1 & 1 \\
1 & 2 & 3 & 3 & 2 & 1 & 1 & -1 & -2 & -1 & 1 & 2 & 2 & 1 & -1 & -2 & -1 & 1 & -1 & -2 & -3 & -3 & -2 & -1 \\
3 & 0 & 0 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & -3 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & -3 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 \\
0 & 3 & 0 & 0 & -3 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 3 & 0 & 0 & -3 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 6 & -6 & -3 & -6 & 6 & -6 & -3 & -6 & 6 & 3 & 6 & 0 & 0 & 3 & 0 & 0 & -3 & 3 & 0 & 0 & -3 & 0 & 0 \\
-6 & 6 & 3 & 6 & -6 & -3 & 3 & 0 & 0 & -3 & 0 & 0 & -6 & -3 & -6 & 6 & 3 & 6 & 0 & 0 & 3 & 0 & 0 & -3 \\
6 & 3 & 6 & -6 & -3 & -6 & 0 & 0 & -3 & 0 & 0 & 3 & -3 & 0 & 0 & 3 & 0 & 0 & 6 & 3 & 6 & -6 & -3 & -6 \\
0 & -3 & 0 & 0 & 3 & 0 & -6 & 6 & 3 & 6 & -6 & -3 & 3 & 6 & -6 & -3 & -6 & 6 & 0 & -3 & 0 & 0 & 3 & 0 \\
3 & 0 & 0 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & -6 & 6 & 3 & 6 & -6 & -3 & 3 & 6 & -6 & -3 & -6 & 6 \\
0 & 0 & 3 & 0 & 0 & -3 & 3 & 6 & -6 & -3 & -6 & 6 & 0 & -3 & 0 & 0 & 3 & 0 & -6 & 6 & 3 & 6 & -6 & -3
\end{vmatrix}$$

Proof. This involves finding the matrix inverse. □

To illustrate, the unanimity type-one profile (see Eq. 7.2) has the decomposition given by the first column of this matrix. To use this matrix to derive other conclusions, see the use of the 6 × 6 corresponding matrix described for three-candidates in Saari [16]. Finding a similar matrix for five (which would be 120 × 120) or six candidates (720 × 720) is possible, but not useful. While I will explain elsewhere how to determine the decomposition of $\mathbf{p}$ for $n \geq 5$ alternatives, it involves a scalar product of $\mathbf{p}$ with the specified the differentials.

8 More structure; many more possible theorems

To conclude, I identify the rich structure of the departure profiles available to discover many new conclusions. To explain, emphasizing the specific departure profiles which change the $\mathbf{d}^k$ outcome for a set is similar to emphasizing the profiles changing pairwise outcome (which is the "departure vector" $\mathbf{d}^2$) for a pair. But focussing on individual sets makes it is difficult to compare what happens among a class of sets. Indeed, about the best we can say is that any ranking is possible.

The pairwise voting results of Saari [15] are a direct consequence of collecting profiles which influence pairwise outcomes into the appropriate Basic and Condorcet differentials. This allows us to identify and compare which profile portions provide consistency in election rankings with which portions cause cycles and other non-transitive outcomes *over different pairs*. It also allows us to understand how the Condorcet portion affects the rankings of k-candidate subsets of candidates, $2 < k < n$. A remarkably similar approach of identifying the appropriate collection of Departure differentials can be used to analyze the behavior of the positional rankings over all k-candidate subsets and how these outcomes affect the rankings of subsets with larger numbers of candidates. As consequences and new conclusions follow the lead and analysis of Saari [15], I emphasize the profile division.

Represent a ranking r of the n-candidates with a ranking disk as described in [15] to define the Condorcet profile. Define the *k-tuple cycle determined by r* to be a list of n sets of k candidates obtained in the following manner. The first k-candidate subset, and its ranking, is given by the first k candidates and their relative ranking in r. Move the ranking disk clockwise so that digit "1" is under the next candidate; the second k-candidate subset, and its ranking, are the k-top ranked candidates in this new ranking. Continue n times until no new subsets are possible.

Definition 9. *For all k-candidate subsets, the c_j-(Basic-Departure) profile differential is*

$$\mathbf{BD}^k_{c_j} = \sum_{c_j \in S, |S|=k} \mathbf{D}^k_{c_j,S}. \tag{8.1}$$

Namely, $\mathbf{BD}^k_{c_j}$ is the sum of all k-candidate departure profiles emphasizing c_j.

For a ranking r of the n candidates, the (r,j)-(Condorcet-Departure) profile differential, $\mathbf{CD}^k_{r,j}$, is sum of all $\mathbf{D}^k_{c_i,S}$ where S is a k-candidate subset in the k-tuple cycle defined by r and $c_i \in S$ is the jth ranked candidate in this ranking.

To illustrate with $k = 3, n = 4$, the (Basic-Departure) differential for A is

$$\mathbf{BD}^3_A = \mathbf{D}^3_{A,\{A,B,C\}} + \mathbf{D}^3_{A,\{A,B,D\}} + \mathbf{D}^3_{A,\{A,C,D\}}$$

while $r = A \succ B \succ D \succ C$ defines the (Condorcet-Departure) profile

$$\mathbf{CD}^k_{r,2} = \mathbf{D}^3_{B,\{A,B,D\}} + \mathbf{D}^3_{D,\{B,D,C\}} + \mathbf{D}^3_{C,\{D,C,A\}} + \mathbf{D}^3_{A,\{C,A,B\}}.$$

These (Basic-Departure) and (Condorcet-Departure) profile differentials play an parallel role for departure outcomes as, respectively, the Basic and Condorcet profiles do for pairwise rankings. As such, most results from [15] have an analogue for k-candidate sets; this includes the interpretation that the (Condorcet-Departure) differentials dismiss information about the individual rationality of voters. (As all statements are in terms of $\mathbf{d}^k$ and $g_{k+s}(\mathbf{d}^k)$ tallies, I show how to translate this to positional outcomes.) The following sample states the the (Basic-Departure) differentials have the same nice properties (with $\mathbf{d}^k$ and $g_{k+s}(\mathbf{d}^k)$) as the Basic profiles. Similarly, the (Condorcet-Departure) differentials, just like the Condorcet differentials, are responsible for all $\mathbf{d}^k$ outcomes which are inconsistent; they cause all cycles and other intransitive behavior.

Theorem 15. *For n candidates, assume* $2 < k < n$ *and that* $1 \leq s \leq (n-k)$.

1. *The* $(k-1)\binom{n}{k}$ *dimensional-subspace of* k*-candidate departure differentials spanned by* $\{\mathbf{D}^k_{c_j,S}\}_{|S|=k,j=1,\ldots,n}$ *also is spanned by the set of* k*-candidate (Basic-Departure) and (Condorcet-Departure) profiles. The (Basic-Departure) differentials define a* $(n-1)$ *dimensional subspace where* $\sum_{j=1}^n \mathbf{BD}^k_{c_j} = \mathbf{0}$.
2. *The* $\mathbf{d}^k$ *and* $g_{k+s}(\mathbf{d}^k)$ *rankings of profile* $\sum_{j=1}^n A^k_j \mathbf{BD}^k_{c_j}$ *agree over all subsets of* k *or more candidates. In particular, the ranking of the candidates for any subset is determined by the ranking of the* $\{A^k_j\}$ *coefficients. The difference between tallies of different candidates in a subset is a fixed multiple times the difference between the coefficients.*
3. *The* $\mathbf{d}^k$ *rankings of* $\mathbf{CD}^k_{r,j}$ *define a cyclic behavior in that each of the* n *candidates is top-ranked in precisely one* k*-candidate subset. More precisely, in the* k*-tuple cycle defined by* r, c_i *is top-ranked in the one* k*-candidate subset where she is* j*th ranked.*
4. *The* $g_n(\mathbf{d}^k)$ *tally of a* k*-candidate (Condorcet-Departure) differential assigns zero points to each candidate.*
5. *The* $g_s(\mathbf{d}^k)$, $k < s < n$, *tally of* $\mathbf{CD}^k_{c_j}$ *need not be a tie.*

Supporting arguments for these assertions closely resemble those for the binary case. For instance, it is shown in [15] that the BC and other procedures change rankings when candidates are dropped because dropping a candidate converts a n-candidate Condorcet profile (which has no effect on positional methods) into a $(n-1)$-candidate Condorcet profile *plus* a profile component in a $(n-1)$-candidate Basic direction; the new Basic component affects the $(n-1)$-candidate positional outcomes. Similarly, the (Condorcet-Departure) profile $\mathbf{CD}^k_{r,j}$, $k < n-1$, has no effect upon $g_n(\mathbf{d}^k)$ (or any method based on this term), but dropping a candidate converts n-candidate $\mathbf{CD}^k_{r,j}$ into $(n-1)$-candidate $\mathbf{CD}^k_{r_1,j}$, where r_1 is the ranking obtained by dropping the candidate from r, *plus* a component in the (Basic-Departure) direction; this new component affects $g_{n-1}(\mathbf{d}^k)$ outcomes (and any procedure using this term). Therefore, $g_s(\mathbf{d}^k)$, $k < s < n$, tally of $\mathbf{CD}^k_{c_j}$ need not be a tie when restricted to s-candidate subsets. In turn, *any procedure based upon* k*-candidate outcomes suffers the problem of having outcomes change when candidates are dropped.*

8.1 Refined division

Another example of the parallels between the pairwise theory of [15] and what can be done with the departure vectors is given next.

Corollary 8. *Suppose a profile consists of k-candidate (Basic-Departure) and (Condorcet-Departure) profiles. A candidate who is $\mathbf{d}^k$ top-ranked in all k-candidate subsets cannot be bottom-ranked in the $g(\mathbf{d}^k)$ election. Similarly, a candidate who is $\mathbf{d}^k$ bottom-ranked in all k-candidate subsets cannot be top-ranked in the $g(\mathbf{d}^k)$ election.*

To add significance to this corollary, and its several extensions, notice that the departure vectors and profiles, along with Theorem 15, provide a nice picture of positional methods and how their rankings are related to what occurs for subsets. I start by describing a different coordinate system for the pyramid $\mathscr{P}^n$ and its derived set $\mathscr{D}^n$.

The center point is $\mathbf{b}^n$. Passing through this point is the axis direction $g_n(\mathbf{d}^3)$; it consists of all $\mathbf{w}^n$ related to the three-candidate positional procedures. The next direction is defined by $g_n(\mathbf{d}^4)$; it identifies the $\mathbf{w}^n$ defined by the four-candidate positional procedures of the form $\mathbf{b}^4 + \lambda\mathbf{d}^4$.. The process continues with axes defined by $g_n(\mathbf{d}^5), \dots, g_n(\mathbf{d}^{n-1})$. These terms complete the coordinate description of $\mathscr{D}^n$. The $\mathscr{P}^n$ system only needs the remaining direction $\mathbf{d}^n$.

The close relationship among the tallies of different sets comes from observing that $\mathscr{P}^n$ coordinate system requires each $\mathbf{w}^n$ to have a unique representation

$$\mathbf{w}^n = \lambda_2\mathbf{b}^n + \sum_{j=3}^{n-1} \lambda_j g_n(\mathbf{d}^j) + \lambda_n \mathbf{d}^n. \tag{8.2}$$

To use Eq, 8.2 to understand the $\mathbf{w}^n$ elections and the new kinds of relationships it admits, notice that the k-candidate (Basic-Departure) differentials not only affect the $g_n(\mathbf{d}^k)$ outcomes, but they agree with the $\mathbf{d}^k$ outcomes (Theorem 15). This provides consistency in election outcomes of Eq. 8.2; it even identifies profile restrictions that provide added relationships. What disrupts consistency, however, is that the k-candidate (Condorcet-Departure) terms have no affect upon the $g_n(\mathbf{d}^k)$ outcomes, but they can change (up to limits of the type specified by Corollary 8) the $\mathbf{d}^k$ outcomes. *This is the source of all inconsistencies in election outcomes.* (When we consider the election tallies of a procedure with fewer candidates, we also have to include the effects of how a (Condorcet-Departure) profile splits into a (Condorcet-Departure) and a (Basic-Departure) profile.)

So, by use of Eq, 8.2 and Theorem 15, we now can, for instance, identify choices of profiles which will allow consistency in election rankings of different sets, we can identify the profiles causing all inconsistencies, we can create all possible paradoxes. This powerful tool will be explored in more detail elsewhere.

8.2 Summary

In summary, we now can explain all differences in outcomes coming from any positional procedures and/or from methods based on pairwise and positional outcomes. The analysis starts with the Basic profiles where there is agreement. Any disagreement in pairwise outcomes can be completely explained by the Condorcet portion of a profile. Any disagreement in the positional rankings of any other subset is based on deviation profile affects. Of equal interest, we can construct examples to illustrate any admissible behavior. This statement holds for all methods based on positional and pairwise outcomes.

9 Proofs

Proof of Theorem 2. All that remains to be proved in Theorem 2 is the assertion that $\mathbf{W}^n$ is a vector in a $\nu(n) = 2^{n-1}(n-4) + n + 2$ dimensional Euclidean space. To derive $\nu(n)$, notice that a j-candidate voting vector has j components where the first is unity and the last is zero. This leaves $j-2$ weights free to be chosen, so the number of free variables in $\mathbf{W}^n$ is $\nu(n) = \sum_{j=2}^{n} \binom{n}{j} (j-2)$. By differentiating the binomial expression

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \tag{9.1}$$

we obtain

$$n(1+x)^{n-1} = \sum_{j=1}^{n} j \binom{n}{j} x^{j-1}. \tag{9.2}$$

After setting $x = 1$ in Eqs. 9.1, 9.2 and using some algebra, the expression for $\nu(n)$ follows. □

Proof of Theorem 3. To prove that the $\mathbf{u}_j^k$ vectors have the desired form, it suffices to consider how many points a voter with preferences $c_1 \succ c_2 \succ \cdots \succ c_k$ casts for each candidate when all $(k-1)$ candidate elections are tallied with $\mathbf{v}_j^{k-1}$. In any subset where candidate c_i, $i \leq j$, is included, c_i receives one point. To count the number of these subsets, notice that each $(k-1)$-candidate subset can be characterized in terms of the missing candidate; each candidate is missing from precisely one set. As this requires c_i to be in all but one subset (that is, she is in precisely $k-1$ sets), she receives $k-1$ points.

Candidate c_{j+1} receives a point only if one of the top j ranked candidates is not present. As each candidate is absent from precisely one subset, c_{j+1} receives a point in j subsets for a total of j points.

It remains to consider c_i where $i > j+1$. For c_i to receive any points, at least two candidates ranked above her must be missing from a $(k-1)$ candidate subset. As this never can occur, c_i receives zero points. This proves that the $\mathbf{u}_j^k$ vectors have the indicated form.

To prove $\mathbf{d}^k$ is a normal vector for $\mathscr{D}^k$, it suffices to prove that $\mathbf{d}^k$ is orthogonal to $\mathbf{u}_j^k - \mathbf{b}^k$. (These vectors span $\mathscr{D}^k$.) As $\mathbf{B}^k = (k-1)\mathbf{b}^k$, an equivalent problem is to show that

$$\mathbf{d}^k \cdot (\mathbf{u}_j^k - \mathbf{b}^k) = \mathbf{d}^k \cdot ((k-1)\mathbf{u}_j^k - \mathbf{B}^k) = 0. \tag{9.3}$$

By use of the equality $\binom{n}{j} = \binom{n}{n-j}$, we have that

$$\mathbf{d}^k \cdot \mathbf{B}^k = \sum_{j=1}^{k-2}(-1)^j \binom{k-1}{j} j.$$

By comparing this expression with the summation obtained from Eq. 9.2 when $n = k-1$ and $x = -1$, it follows that

$$\mathbf{B}^k \cdot \mathbf{d}^k = \sum_{j=1}^{k-2}(-1)^j j \binom{k-1}{j} = (k-1)(1-1)^{k-2} + (k-1) = k-1 \tag{9.4}$$

It remains to use induction to prove that $(k-1)\mathbf{u}_j^k \cdot \mathbf{d}^k = k-1$. Only the second coordinate of both $(k-1)\mathbf{u}_1^k$ and $\mathbf{d}^k$ are nonzero; they are, respectively 1 and $k-1$. It follows that $(k-1)\mathbf{u}_1^k \cdot \mathbf{d}^k = k-1$.

Using the induction hypothesis, assume that $(k-1)\mathbf{u}_j^k \cdot \mathbf{d}^k = k-1$ for $j < s$. We need to establish that $(k-1)\mathbf{u}_s^k \cdot \mathbf{d}^k = k-1$, or that $[(k-1)\mathbf{u}_s^k - (k-1)\mathbf{u}_{s-1}^k] \cdot \mathbf{d}^k = 0$.

The vector in the bracket has non-zero values only in the sth and $(s+1)$th position; they are, respectively, $k-s$ and s. In turn, this means that the scalar product becomes (up to a sign) $(k-s)\binom{k-1}{k-s} - s\binom{k-1}{k-(s+1)}$. Using the binomial expressions, this becomes

$$\begin{aligned}(k-s)&\frac{(k-1)!}{(k-s)!((k-1)-(k-s))!} - s\frac{(k-1)!}{(k-1-s)!s!} \\ &= (k-1)![\frac{1}{(k-s-1)!(s-1)!} - \frac{1}{(k-s-1)!(s-1)!}] = 0.\end{aligned} \tag{9.5}$$

This completes the induction proof. □

Proof of Theorem 4. The linearity of the tallying procedure ensures there is a kernel. The fact the universal kernel, determined by the pairwise and plurality votes, is contained in the kernel of all procedures is a direct consequence of Corollary 4. It remains is to find the dimension which, from the linearity of the tallying procedure, is the difference between the dimensions of the normalized space of profiles $(n!-1)$ and the the normalized space of vote tallies.

The normalized space of vote tallies is where, instead of election tallies, we compute the fraction of the total vote received by each candidate. As a k candidate election has $(k-1)$ degrees of freedom, the dimension of all pairwise elections is $\binom{n}{2}(2-1)$, of all triplets is $\binom{n}{3}(3-1), \ldots$. The total dimension is $\sum_{j=2}^{n}(j-1)\binom{n}{j} = \sum_{j=2}^{n} j\binom{n}{j} - \sum_{j=2}^{n}\binom{n}{j}$.. It follows from Theorem 1 that this is the dimension of the plurality vote image space. To compute this summation and show that the dimension of the kernel is as specified is a straightforward computation that uses Eqs. 9.1, 9.2 in a manner similar to the derivation of $\nu(n)$. □

Proof of Theorem 6.
Part 1. In $\mathbf{C}^n_r$, each candidate is in each position twice; once with one voter and once with (-1) voters. Thus, $\mathbf{d}^n$ tally must be zero. If $k < n$ is odd, then $\mathbf{d}^k$ has the same form that the jth coefficient equals the $(k+1-j)$ coefficient. Each ranking $r \in \mathbf{C}^n_r$ has $\rho(r) \in \mathbf{C}^n_r$. When restricted to set S, if a candidate is jth ranked in r, then she is $(k+1-j)$th ranked in $\rho(r)$. The cancellation of votes proves the assertion. For even values of $k < n$, the jth coefficient of $\mathbf{d}^k$ is the negative of the $(k+1-j)$th coefficient, so, rather than leading to a cancellation, the above $r, \rho(r)$ symmetry adds to a candidate's tally. With this observation, examples now are easy to construct.

Part 2. As each $\mathbf{C}^n_r$ is orthogonal to each Basic profile, so are the $c_i \succ c_j$-Condorcet profiles. Because each $\mathbf{C}^n_r$ has a zero $\mathbf{d}^k$ outcome when $k = n$ or when k is odd, the same assertion holds for the $c_i \succ c_j$ Condorcet differentials. It remains to prove the assertion when k is even.

To prove the assertion about the $A \succ B$ Condorcet profile differential for k even, let $\mathscr{S}$ be a k-candidate subset. If A and B are not in $\mathscr{S}$, then because we are dealing with a profile differential where candidates other than $\{A, B\}$ are treated symmetrically, my standard argument shows that all candidates receive a zero $\mathbf{d}^k$ tally. If both candidates are in $\mathscr{S}$, then to determine the A tally, we need to determine how often A is jth ranked in $\mathscr{S}$. In the $A \succ B$ Condorcet profile differentials, the only condition on these rankings is that A is ranked immediately above B; all such rankings are included. The important point is that the number of rankings with this property in a $A \succ B$ Condorcet profile differential is the same for all j satisfying the relevant values of $2 \leq j \leq k-1$. (It is not necessary to determine when A is top or bottom ranked in $\mathscr{S}$ because the corresponding $\mathbf{d}^k$ coefficient is zero.)

To see this, notice that once the slots for the $\mathscr{S}$ candidates within the n possible positions are determined, we need to compute the number of ways to rank the $\mathscr{S}$ candidates. Of these, there are $\binom{k-2}{j-1}$ ways to select which candidates are ranked above A, and each choice can be ranked in $(j-1)!$ ways. The number of ways to rank the candidates below B is $(k-2-(j-1))!$. The product, which gives the total number of such rankings, is $(k-2)!$. It remains to determine how many ways to select the rankings for k positions within n positions where two slots are together in the jth and $(j+1)$th positions (to accommodate the adjacent A, B ranking. Using standard combinatoric approaches, where the adjacent rankings are treated as one unit, this is $\binom{n-1}{k-1}$. Therefore the total number of rankings where A is in jth position is $\binom{n-1}{k-1}(k-2)!$. Of more value than the actual number is that it does not depend upon j. Consequently, A is ranked in jth position as often as in $(k-j)$th position. But, as these $\mathbf{d}^k$ coefficients differ only in sign, the terms cancel. A similar argument holds for the $\rho(r)$ portion of each profile, and for B. Therefore the A and B tallies with the departure vector $\mathbf{d}^k$ are zero. By using my standard symmetry argument, the same assertion extends to all candidates.

If only one of A or B is in $\mathscr{S}$, then the only minor changes in the above argument show that the candidate is ranked in jth place as often as she is ranked $(k-j)$th place. This gives the same argument to prove that the $\mathbf{d}^k$ tally is zero.

Since the departure vectors give a zero tally for all $c_i \succ c_j$ Condorcet and Basic profiles, a positional procedure (with Eq. 8.2 representation) is determined strictly by the BC outcome for these profiles. This completes the proof of this part.

Part 3. This is a direct tally. In $\{c_i, c_j\}$ tallies, we know from [15] the tallies for each candidate for each $\mathbf{C}^n_r$ component. The conclusion follows by computing the number of Condorcet differentials in the $c_i \succ c_j$ Condorcet differential. For any pair of candidates where neither is c_i or c_j, the symmetric manner in which they are treated in the $c_i \succ c_j$ Condorcet profile requires a tie election. As this is a profile differential, each candidate receives zero points.

Finally, the assertion for $\{c_i, c_s\}$ and $\{c_j, c_s\}$ elections can be determined by a direct computation, or by observing that each candidate c_s, $s \neq i, j$, must be treated symmetrically. Furthermore, since we are dealing with a profile differential, the sum of all pairwise tallies has to equal zero. The conclusion now follows.

Part 4. From part 2, we know that the positional tally must agree with the BC tally. The rest of the assertion is a direct computation. □

Proof of Theorem 7. Each $\mathscr{E}^n_{k,j}$ can be expressed as the profile $T^n_{k,j}$ where n voters are assigned to each ranking where c_j is kth ranked *minus* $\mathbf{K}^n$. Part 3 now follows from a direct computation.

As $\sum_{j=1}^n \mathscr{E}^n_{k,j} = \sum_{j=1}^n T^n_{k,j} - n\mathbf{K}^n$,the second summation has each of the $n!$ rankings represented n times. This equals $n\mathbf{K}^n$ so $\sum_{j=1}^n \mathscr{E}^n_{k,j} = \mathbf{0}$. (This is part of Part 2). Thus, the set $\{\mathscr{E}^n_{k,j}\}_{j=1}^n$ can span at most a $(n-1)$-dimensional space. That it spans a space of this dimension follows from the form of the $T^n_{k,j}$ terms; any two have no rankings in common.

Part 1. As any differential $\mathbf{p}$ is orthogonal to $\mathbf{K}^n$, if $\mathbf{p}$ is orthogonal to the space spanned by $\{\mathscr{E}^n_{k,j}\}_{j=1}^n$, then $\mathbf{p}$ is orthogonal to each $T^n_{k,j}$. This means that $\mathbf{p}$ has an equal number of rankings with positive and with negative number of voters where c_j is kth ranked. As this is true for all c_j, the $\mathbf{E}^n_j$ tally of $\mathbf{p}$ must be a complete tie with zero tallies.

Part 4. Without loss of generality, for this part consider $\mathscr{E}^n_{k,1} = T^n_{k,1} - \mathbf{K}^n$. The symmetry of $\mathbf{K}^n$ requires each pair to end in a $n!/2 : n!/2$ tie. Therefore, the pairwise outcomes are determined by $T^n_{k,1}$. To compute the $T^n_{k,i}$ tally for pair $\{c_i, c_s\}$, $i, s \neq 1$, notice that for each ranking $r \in T^n_{k,1}$ there is a ranking $r' \in T^n_{k,1}$ where the only difference between r and r' is that the positions of c_i, c_s are interchanged. This means each candidate gets the same pairwise vote of $n(n-1)!/2$. The conclusion that the $\{c_i, c_s\}$ vote is a zero-zero tie now follows.

Now consider the tally of $\{c_1, c_s\}$. For each $r \in T^n_{k,1}$, c_1 must be in kth position while c_s can be in any position. For all rankings where c_s is ranked above c_1, c_s gets the vote. As there are $(k-1)$ such positions and as there are $(n-2)!$ rankings with c_1, c_s in these specified positions, c_s receives $n(k-1)((n-2)!)$ points. There are $(n-k)$ positions to rank c_s below c_1, and each is accompanied by the $(n-2)!$ ways to rank the remaining candidates, so c_1 receives $n(n-k)((n-2)!)$ votes. The contribution to each candidate from $-\mathbf{K}^n$ is $-n!/2$ votes. This leads to the specified election tally.

As each Condorcet profile differential has c_1 in kth position in precisely two rankings where one has one voter and the other has (−1 voters, it follows that $\mathscr{E}^n_{k,1}$ is orthogonal to all Condorcet profiles. But, since the pairwise votes are determined by the Basic and Condorcet profiles and since they are not all zero, it follows that $\mathscr{E}^n_{k,1}$ is not orthogonal to some Basic profile.

To show the final result, it suffices to show for $j \neq k$ that the scalar product

$$(\mathscr{E}^n_{k,1}, \mathscr{E}^n_{j,1}) = (T^n_{k,1}, T^n_{j,1}) - (\mathbf{K}^n, \mathscr{E}^n_{j,1}) - (T^n_{k,1}, \mathbf{K}^n) \neq 0.$$

The first term on the right side is zero because they have no rankings in common, the second term is zero because $\mathbf{K}^n$ is orthogonal to all profile differentials. For the last term, each $r \in T^n_{k,1}$ occurs in $\mathbf{K}^n$ so the scalar product is $n(n-1)! \neq 0$ completing the proof. □

Proof of Theorem 8.

Part 1. To prove that the pairwise tallies are zero, recall that a profile consisting of the two rankings r and $\rho(r)$ forces a tie vote for all pairs. As $\mathbf{R}_{k,j}$ consists of appropriate $(r, \rho(r))$ pairs, we are assured that all pairwise tallies are zero. The terms with negative numbers of voters are inserted only to make $\mathbf{S}_{k,j}$ a profile differential.

An alternative way to define $\mathbf{A}_{k,j}$ is to start with two copies of a ranking $r*$ of the $(n-1)$ candidates other than c_j. Insert c_j into the first copy so she is kth position, and into the second copy so she is in $(n+1-k)$th position. Assign one voter to the first augmented ranking, −1 to the second one. With two other copies of $r*$, insert c_j into them so, respectively, she is top and bottom-ranked. Assign x voters to the first augmented ranking, $-x$ to the second one where the value of x is to be determined. Clearly, $\mathbf{A}_{k,j}$ is the sum of this set of four rankings over all choices of $r*$. By construction, $\mathbf{A}_{k,j}$ is a profile differential.

First consider $\{c_i, c_s\}$ pairs where neither candidate is c_j. In each of the two pairs created by $r*$, the $\{c_i, c_s\}$ ranking remains fixed, but the number of voters have different signs, so the pairwise tallies equal zero. In a $\{c_j, c_s\}$ pairwise election, c_j beats c_s in each ranking where c_j is top-ranked; as there are $(n-1)!$ such rankings and as each ranking occurs x times, c_j receives $x(n-1)!$ votes. When c_j is bottom ranked, she never beats c_s. The number of times c_j beats c_s when c_j is kth ranked in a $\mathbf{A}_{k,j}$ ranking equals the number of rankings where c_s can be ranked below c_j; this is $\binom{n-2}{k-1}(k-1)!(n-k)!$. Similarly, the number of rankings where c_j is ranked above c_s when c_j is $(n+1-k)$th ranked is $\binom{n-2}{k-2}k!(n-k)!$. Therefore, to have a pairwise tie, we need to choose x so that c_j's tally for $\mathbf{A}_{k,j}$ is zero; that is, we need to solve

$$x(n-1)! + [\binom{n-2}{k-1} - \binom{n-2}{k-2}]k!(n-1)! = 0.$$

This is $x = -(n+1-2k)(n-1)$.

Part 2. First, let $k \neq (n+1)/2$. The $\mathbf{E}_k + \mathbf{E}_{n+1-k}$ tally for c_j for each $(r, \rho(r))$ where c_j is kth ranked in r is 2. As there are $(n-1)!$ such rankings in $\mathbf{S}_{k,j}$, and as c_j is never kth or $(n+1-k)$th ranked in the remaining rankings, c_j's tally is $2(n-1)!$. When $k = (n+1)/2$, then $\mathbf{E}_k + \mathbf{E}_{n+1-k} = 2\mathbf{E}_k$. In this setting, c_j is middle ranked in $(n-1)!$ of the $\mathbf{S}_{k,j}$ rankings, so again she receives $2(n-1)!$ votes. As $\mathbf{S}_{k,j}$ is a profile differential, the sum of the $\mathbf{E}_k + \mathbf{E}_{n+1-k}$ tallies over all candidates must equal zero. As all remaining candidates are treated symmetrically, each of these candidates receives the same number of votes. Therefore, each remaining candidate receive $-2(n-2)!$ votes.

The $\mathbf{E}_s + \mathbf{E}_{n+1-s}$ tally of $\mathbf{S}_{k,j}$, $s \neq k$, counts the number of times each candidate is in sth and in $(n+1-s)$th position. As c_j never is in any of these positions, her tally is zero. Again, as $\mathbf{R}_{k,j}$ is a profile differential and as the other candidates are treated symmetrically, each remaining candidate receives a zero tally.

By definition, each ranking r in $\mathbf{A}_{k,j}$ is accompanied with its reversal $\rho(r)$ but with a negative number of voters. Therefore, the $\mathbf{E}_s + \mathbf{E}_{n+1-s}$ tally of a candidate for this $r, \rho(r)$ pair is zero. Hence, for $\mathbf{A}_{k,j}$, the $\mathbf{E}_s + \mathbf{E}_{n+1-s}$ tally for each candidate is zero; $s = 2, \ldots, (n-1)/2$.

Part 3. For c_j, the $\mathbf{E}_k - \mathbf{E}_{n+1-k}$ tally of $\mathbf{A}_{k,j}$ is $2(n-1)!$. Since $\mathbf{A}_{k,j}$ is a profile differential and since each of the remaining candidates is treated symmetrically, each remaining candidate receives $-2(n-2)!$ votes. For each r in $\mathbf{S}_{k,j}$, $\rho(r)$ appears with an equal number of voters, so the $\mathbf{E}_s - \mathbf{E}_{n+1-s}$ tally is zero for each candidate. Thus, the $\mathbf{E}_s - \mathbf{E}_{n+1-s}$ tally of $\mathbf{S}_{k,j}$ assigns zero points to each candidate.

The $\mathbf{E}_s - \mathbf{E}_{n+1-s}$ tally for c_j with $\mathbf{A}_{k,j}$, $s \neq k$, equals the number of times c_j is in sth place minus the number of times she is in $(n+1-s)$th place. But c_j never has such a ranking in $\mathbf{A}_{k,j}$, so her tally is zero. Again, because $\mathbf{A}_{k,j}$ is a profile differential and because each remaining candidate is treated symmetrically, the tally for each candidate is zero.

Part 4. This is a direct consequence of the $(\mathbf{E}_k \pm \mathbf{E}_{n+1-k})$ tallies from parts 2, 3. □

Proof of Theorem 9. For $n \geq 4$, there are $(n-2)$ choices of $\mathbf{E}^n_k$ vectors. A ranking is assigned for each choice in the following manner. Partition the candidates into two sets with an equal number of candidates if n is even, and differ by one candidate otherwise. For instance, the first set might have the first $n/2$ or $(n+1)/2$ candidates, and the second has the rest. Define a Condorcet cycle for the candidates from each set (the cycle has as many rankings as candidates in the set). For $n \geq 4$, the number of rankings for either set does not exceed the number of $\mathbf{E}^n_k$ variables. So, for each $k = 2, \dots, (n+1)/2$, choose a ranking from each Condorcet cycle to define the $\mathbf{E}^n_k$ ranking; for n odd, there is one repetition. (So, for $n = 5$, the three rankings could be $(c_1 \succ c_2 \succ c_3) \succ (c_4 \succ c_5), (c_2 \succ c_3 \succ c_1) \succ (c_5 \succ c_4), (c_3 \succ c_1 \succ c_2) \succ (c_4 \succ c_5)$.)

For each k, use the Positional differentials associated with $\mathbf{E}^n_k$ can be used to define a profile establishing the assigned ranking. According to Eq. 3.3, if α_k is the only non-zero coefficient, then the ranking is the assigned one if $\alpha_k > 0$, and the reverse of this for negative α_k values. For even values of n, this establishes the conclusion.

For odd values of n, it remains to show that each candidate in the smaller set can be middle ranked. Toward this end, notice that the $\mathbf{E}^n_k$ tally of c_j, $t^n_{k,j}$, is strictly determined by the $\{\mathbf{P}^n_{k,j}\}^n_{j=1}$ coefficients; the only constraint is that $\sum_{j=1}^n t^n_{k,j} = 0$. (Thus, $t^n_{k,n} = -\sum_{j=1}^{n-1} t^n_{k,j}$.) In particular, $\mathbf{E}^n_k$ the tally is independent of those for other $\mathbf{E}^n_s$. Thus, viewing the tallies as a linear mapping from the space of coefficients to the $(n-2)(n-1)$ dimensional space of tallies, it is a open mapping.

A ranking changes with the procedure because pairs change rankings by passing through the $\{\alpha_k\}_{k=2}^{n-1}$ values which define a tie. Each candidate is top and bottom ranked with some procedure, so the assertion follows if it always is possible to change the rankings one pair at a time. If the claim is false, then it must be that some candidate, say D, always is moved from a position right below middle ranked to a position above middle ranked. Using the transitivity of positional rankings, this requires at least a triple, say $\{B, C, D\}$, so that the same α_k values define the three rankings $B \sim C, C \sim D, B \sim D$. (This occurs with $[2\mathbf{P}^4_{2,A} + \mathbf{P}^4_{2,B} - \mathbf{P}^4_{2,C}] + [2\mathbf{P}^4_{3,B} + \mathbf{P}^4_{3,D} - \mathbf{P}^4_{3,A}]$ along the line $\alpha_2 = -\alpha_3$.) I now show this cannot occur in general.

For a given tally, the procedures causing an indifference between a pair, say B, D, are given by those α_k values satisfying

$$\sum_{k=2}^{n-2} t^n_{k,B} = \sum_{k=2}^{n-1} t^n_{k,D}.$$

Therefore, for a simultaneous triple to have tied outcomes involving, say, c_1, it must be that the $(n-2) \times (n-2)$ matrix

$$\begin{pmatrix} t^n_{1,1} - t^n_{1,2} & \dots & t^n_{(n-1),1} - t^n_{(n-1),2} \\ \dots & \dots & \dots \\ t^n_{1.1} - t^n_{1,(n-1)} & \dots & t^n_{(n-2),1} - t^n_{(n-2),(n-1)} \end{pmatrix}$$

has lower rank. As this is a lower rank condition, it follows from the open mapping condition that coefficients can be found to avoid this behavior. This completes the proof.

To prove the second part, recall that the pairwise rankings can be selected to satisfy any desired arrangement by using only Basic and Condorcet profiles. These rankings only affect the BC outcome. Similarly, the Positional differentials have no effect upon the pairwise outcomes. So, by choosing the Positional differentials to have sufficiently large tallies, the BC outcome has minimal effect upon the rankings of the other procedures. The conclusion now follows. □

Proof of Theorem 11.

Part 1. We compute the $\{c_i, c_j\}$ pairwise tally. If $c_i \in \mathscr{S}$ but c_j is not, then $c_i \succ c_j$ in each ranking in $\mathbf{D}^n_{c_i,\mathscr{S}}$. As the number of rankings which has c_i in sth spot is $(k-1)!(n-k)!$, the total number of votes c_i wins in the pairwise elections is $(k-1)!(n-k)!\sum_{s=1}^{k}(-1)^s\binom{k-1}{s-1}$. According to Eq. 9.1 where $x = -1$, this summation equals zero.

Now suppose both $c_i, c_j \in \mathscr{S}$. Among all of the $\mathbf{D}^n_{c_i,\mathscr{S}}$ rankings which has c_i in sth position, $(k-s)(k-2)!(n-k)!$ have c_j ranked lower. Therefore, the total number of points earned by c_i in the pairwise vote is a $(k-2)!(n-k)!$ multiple of

$$\sum_{s=1}^{k}(-1)^s(k-s)\binom{k-1}{s-1} = \sum_{s=1}^{k}(-1)^s(k-s)\binom{k-1}{k-s}. \tag{9.6}$$

According to Eq. 9.2 where $x = -1$, this sum is zero. This completes the proof of this part.

Part 2. According to Corollary 4 and part 1, it suffices to show that the plurality votes for all candidates in $\mathscr{S}'$ is a tie. If no candidates from $\mathscr{S}'$ are in $\mathscr{S}$, then each candidate is treated symmetrically. This means that because $\mathbf{D}^n_{c_i,\mathscr{S}}$ is a profile differential, the sum of the votes equals zero and each candidate receives the same vote. Thus, each candidate has a zero plurality tally.

Now suppose there is at least one candidate in $\mathscr{S}$ that is not in $\mathscr{S}'$ and that $\mathscr{S} \cap \mathscr{S}' \neq \emptyset$. The plurality tally of any $\mathscr{S}'$ candidate not in $\mathscr{S}$ is, trivially, zero. If c_i is in $\mathscr{S}' \cap \mathscr{S}$, then her plurality tally is determined by the number of rankings where she is top ranked in $\mathscr{S}$ or where candidates in $\mathscr{S}$ but not in $\mathscr{S}'$ are ranked above her. Suppose there are $s \geq 1$ candidates of this type. This means c_1 is in top-place in $(n-k)!(k-1)!$ rankings, second place in $(n-k)!\binom{s}{1}1!(k-2)!$ rankings, third place in $(n-k)!\binom{s}{2}2!(k-3)!$ rankings, $\ldots$, $(n-k)!\binom{s}{j}j!(k-j-1)!$ rankings in jth place. The plurality vote is determined by the number of voters with each of these rankings, so the tally is $-(n-k)!$ times the value

$$\begin{aligned}
&\sum_{j=0}^{s}(-1)^j\binom{s}{j}j!(k-j-1)!\binom{k-1}{j}\\
&= \sum_{j=0}^{s}(-1)^j\binom{s}{j}j!(k-j-1)!\frac{(k-1)!}{j!(k-j-1)!}\\
&= (k-1)!\sum_{j=0}^{s}(-1)^j\binom{s}{j} = (1-1)^s = 0.
\end{aligned} \tag{9.7}$$

Again, the symmetry for the other voters in $\mathcal{S} \cap \mathcal{S}'$ ensures that their plurality vote is zero. A similar argument holds for $\mathcal{S} \cap \mathcal{S}' \neq \varnothing$ where c_i is not in this set.

Part 3. To prove this assertion, it suffices to prove that all $\mathbf{u}_j^k$ tallies of $\mathbf{D}^n_{c_i,\mathcal{S}}$ in $\mathcal{S}$ are zero. Because there are only two non-zero terms in $(k-1)\mathbf{u}_j^k$, its tally is determined by the number of ways the candidates can be ranked to have c_i is in first and in second place. Because this number is the same (it is $(k-1)!(n-k)!$), we only need to multiply the number of voters times the assigned points. All other candidates from $\mathcal{S}$ are treated symmetrically. This means that the tally is $(n-k)!(k-1)![((k-1)(-1) + \binom{k-1}{1}(1)] = 0$. As $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, the sum of the total vote is zero, and each of the other candidates receives the same tally. Thus, their tally also is zero.

With the induction hypothesis, assume that the c_i tally with $(k-1)\mathbf{u}_j^k$ is zero for $j < s$. We now must show that the tally is zero for $(k-1)\mathbf{u}_s^k$. But this computation is the same as that given in Eq. 9.5. (This reflects the duality of the construction.) The same symmetry argument shows that the tally for the other candidates also is zero.

Part 4. To see that c_i has a positive tally with $\mathbf{d}'$, notice that the sign of the coefficients of $\mathbf{d}'$ and the number of voters for each of the admissible c_i rankings agree. Hence, c_i receives a positive vote. All other candidates are treated symmetrically within the groups $\mathcal{S} - \{c_i\}$ and $\mathcal{S}' - \mathcal{S}$, so, within these groups, they receive the same tally. Suppose there are β more candidates in $\mathcal{S}'$ then in $\mathcal{S}$. The tally for each candidate in $\mathcal{S}' - \mathcal{S}$ is the sum of the last β coefficients of $\mathbf{d}'$. So, if $\beta = 1$, then this sum is zero. For $\beta > 1$, the sign of this total depends upon the parity of $|\mathcal{S}|$ and of β. Because $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, the sum of the total number of points is zero.

Part 5. The $\mathcal{S}$ plurality tally for c_i is determined by the number of times she is top-ranked in $\mathbf{D}^n_{c_i,\mathcal{S}}$. But this is $(k-1)!(n-k)!$. As (-1) voters are assigned to each ranking, the tally is as stated. Each of the other $\mathcal{S}$ candidates is treated symmetrically, so each receives the same plurality tally. But $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, so the sum of the votes equals zero. Therefore, each of these other candidates receives a $(n-k)!(k-2)!$ plurality tally.

Part 6. By use of the symmetry argument in the proof of part 5, it suffices to prove that c_i is $\mathbf{d}^k$ top-ranked in $\mathcal{S}$ with the $\mathbf{D}^n_{c_i,\mathcal{S}}$ profile. But the signs of the jth coefficient of $\mathbf{d}^k$ and the number of voters when c_i is jth ranked agree. Therefore, c_i's tally is $(k-1)!(n-k)! \sum_{j=2}^{k-1} [\binom{k-1}{j}]^2$. □

Part 7. This is a direct computation.

Proof of Theorem 15. Parts 3 and 5 are direct computations. To prove part 4, notice that the $g_{k+1}(\mathbf{d}^k)$ tally for a candidate is the sum of points she receives in $\mathbf{d}^k$ elections. By induction, the $g_n(\mathbf{d}^k)$ outcome is a sum of sums of … of the votes a candidate gets in the various subsets. But, the summetry of the (Condorcet-Departure) profile requires (as with the Condorcet profile in pairwise voting) a complete tie. Similarly, the proof of part 2 also uses the linear properties of the dot product (or tallying process). It follows from Theorem 11 that the number of points each candidate receives in each k-candidate subset is a multiple of the coefficients of $\sum_{j=1}^{n} A_j^k \mathbf{BD}_{c_j}^k$. By definition, the $g_{k+1}(\mathbf{d}^k)$ tally for each candidate c_i is the sum of the number of points she received in the k-candidate elections, so the multiple changes, but it still remains a multiple of the A_j^k terms. The rest of this part follows from induction. □

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