Economic Theory **15**, 1–53 (2000)

Economic Theory

Research Articles

Mathematical structure of voting paradoxes[*]

I. Pairwise votes

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Received: June 22, 1998; revised version: February 14, 1999

Summary. A theory is developed to identify, characterize, and explain all possible positional and pairwise voting outcomes that can occur for any number of alternatives and any profile. This paper describes pairwise voting where new results include explanations for all paradoxes, cycles, conflict between Borda and Condorcet rankings, differences among procedures using pairwise votes (such as the Borda Count, Kemeny's method, and the Arrow-Raynaud rule), and discrepancies among the societal rankings as candidates are dropped or added. Other new results include new relationships among the Borda and Condorcet "winners" and "losers." The theory also shows how to construct all supporting profiles. The following companion paper does the same for positional methods.

Keywords and Phrases: Positional voting, Voting paradoxes, Social choice.

JEL Classification Number: D71.

1 Introduction

Positional procedures are the commonly used voting methods where points are assigned to alternatives according to how each voter positions them on the ballot. Then the candidates are ranked according to the *election tallies* – the sums of assigned points. Familiar choices are the plurality vote which assigns a single

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[*] This research was supported by NSF grant DMI-9971794 and an Arthur and Gladys Pancoe Professorship. Earlier versions were written and results introduced on 1995 and 1996 visits to Université de Caen; my thanks to my host Maurice Salles. A revised version was written at the IMBS at UCI; my thanks to my host Duncan Luce.

point to a voter's top-ranked candidate and zero to all others, and the Borda Count (BC) which assigns $n-1, n-2, \ldots, n-n=0$ points, respectively, to a voter's first, second, ..., nth ranked candidate. Although widely used, the disturbing paradoxes (i.e., counter-intuitive outcomes) these methods admit cause the legitimate worry that we may not always select who or what we really want. Indeed, the academic study of this topic was initiated when Borda [3], and then his fellow mathematicians Condorcet, Laplace, and others, raised this concern about the French Academy elections of the 1780s.

To illustrate what can happen with a four-candidate example, where $A \succ B$ means that A is strictly preferred to B, suppose the transitive preferences of the 30 voters are

Number	Ranking	Number	Ranking
3	$A \succ C \succ D \succ B$	2	$C \succ B \succ D \succ A$
6	$A \succ D \succ C \succ B$	5	$C \succ D \succ B \succ A$
3	$B \succ C \succ D \succ A$	2	$D \succ B \succ C \succ A$
5	$B \succ D \succ C \succ A$	4	$D \succ C \succ B \succ A$

(1.1)

The easily computed $A \succ B \succ C \succ D$ plurality outcome, with a 9:8:7:6 tally, suggests that A is the voters' top-choice. But, is she? If any candidate, or pair of candidates, drops out of contention; the new plurality ranking *reverses* to agree with $D \succ C \succ B \succ A$. (For instance, if C drops out, the new $D \succ B \succ A$ outcome has a 11:10:9 tally. If B and D drop out, the $C \succ A$ outcome has a 21:9 tally.) Rather than A being the candidate of choice, it is arguable that the voters' "correct" choice is D even though she is the plurality bottom-ranked candidate.

1.1 Complexity

Positional procedures have proved to be formidable to analyze; the complexity already is suggested by the Eq. 1.1 changes in societal outcomes when alternatives are dropped. The problems, however, can be much worse; e.g., select a ranking for each of the $2^n - (n+1)$ subsets of alternatives – do so randomly or to deliberately create a perverse set of rankings – and there is a profile (a *profile* lists each voter's preferences) where the voters' sincere plurality outcome for each subset is the selected one (Saari [16]).

Other kinds of difficulties arise by ranking the same alternatives with different methods. For instance, tallying the Eq. 1.1 profile with the positional method $(12, 1, 0, 0)$ (which assigns 12, 1, 0, 0 points, respectively, to a voter's top, second, third, and last ranked candidate) leads to a $B \succ A \succ C \succ D$ ranking, method $(36, 6, 1, 0)$ defines $C \succ A \succ B \succ D$, while the BC ranking of $D \succ C \succ B \succ A$ reverses the plurality outcome. The voters' views are fixed, but *each* candidate "wins" with some procedure. A more extreme example is a ten-candidate profile allowing *millions of different election rankings* (Saari [20]). The voters' opinions remain fixed by the marked ballots, yet varying the choice of a positional method creates millions of contradictory election outcomes where each candidate wins with some procedures but is bottom-ranked with others. This suggests that election outcomes may more accurately reflect the choice of a procedure than the voters' preferences.

A natural consequence of this complexity is Riker's [13] resigned attitude that "[t]he choice of a positional voting method is subjective." Another expert stressed the importance of *social choice* – where the emphasis is on the election winner rather than a ranking – by arguing that "[g]iven all the logical barriers that have to be scaled to even come close to making a coherent social choice, demanding a full ordering is a tall order." Indeed, searching for a full ordering is "something that most of us long ago gave up on as impossible and/or incoherent." His thoughts probably reflect the general sense of the choice community. The difficulties and frustrations of this area are further captured by Arrow's Theorem [1] (see Section 8) which, loosely speaking, asserts that once there are three or more candidates, all procedures are flawed.

In this and a companion paper I address and resolve some of these concerns by developing the mathematical structures which allow us to understand majority vote rankings of pairs of candidates (called *"pairwise voting"* in what follows) and how they are related to the rankings of all positional procedures. The companion paper describes positional methods. While this current paper offers certain new positional voting results, the emphasis is on pairwise voting and its many consequences. This includes developing the strongest possible relationships between pairwise and positional rankings, and showing how certain pairwise outcomes force inconsistencies among positional outcomes. Also, as shown, the approach developed here now allows us to identify, characterize, and explain all paradoxes suffered by any method which uses pairwise votes; it allows us to characterize all profiles supporting any specified outcome; it explains all single profile paradoxes. As the companion paper does the same for positional procedures, the mathematical structures of these papers provide the information needed for the selection and analysis of voting procedures. The generality of these results suggests we should expect surprises; indeed, some conclusions contradict widely accepted beliefs of the field.

2 Approach

Instead of the traditional axiomatic and combinatoric methods, I emphasize the geometry of profile space by decomposing it into subspaces. Each subspace plays a role similar to that of eigenspaces of a matrix in that each subspace has a specific function; their profiles affect only a specified class of procedures while having no impact on other methods. Consequently, *these profiles, and only these profiles,* cause all of the paradoxes which involve these particular procedures. Conversely, *any* specified profile can be fully analyzed because each subspace component determines how this profile affects the associated procedures. A surprise (Section 3.1) is that a surprisingly large portion of profile space consists of profiles with no effect on election outcomes; they just confuse and complicate the analysis. The decomposition given here (Section 3.2) separates the “neutral” components, called *Kernel profiles*, from the “effective” parts.

2.1 The decomposition

A naive, false belief is that the choice of an election procedure does not matter because the outcome is essentially the same no matter which method is used, no matter what candidates are added or dropped. While this belief identifies a tacit but major goal, it is so stringent that it cannot be satisfied even with an unanimity profile. (The three candidate unanimity plurality outcome differs from the BC outcome which differs from the outcome where each voter votes for two candidates.) Indeed, there is no reason to expect these conditions ever to be satisfied.

Surprisingly, (Section 5) there exists a $(n-1)$-dimensional subspace of these desired profiles. They avoid all conflict because the election outcomes *and even the tallies* for all positional procedures always agree even over all subsets of candidates. To honor their special status that these, and only these profiles avoid all possible election conflict, they are called *Basic profiles*. But if the Basic profiles are free from conflict, then all election difficulties are caused by profiles orthogonal to the Basic profile subspace. To emphasize that only these orthogonal profiles force election outcomes to deviate from a desired consistency, they are called *profile deviations*.

So, in order to understand why procedures have conflicting outcomes, we must understand the *space of profile deviations.* To do so, I further decompose this portion of profile space into those subspaces responsible for all conflict in pairwise rankings, for all conflict among the positional rankings of different subsets of candidates, and for all conflict in election outcomes for the same subset of candidates. (The complete description for three alternatives is in (Saari [23].)

The important class of profile deviations introduced in this paper (Section 6) are what I call the *Condorcet* profiles. These profiles, which are responsible for *all* pairwise election differences, are related to the familiar *Condorcet triplet* (Condorcet [5]) where the preferences for three voters are

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B. \tag{2.1}$$

To explain, recall that the pairwise voting cycles have been widely studied since discovered by Condorcet. Condorcet also recognized that profiles of the Eq. 2.1 type cause cycles; this insight has been exploited by others to derive conditions specifying when cycles occur (e.g., a small sample of recent papers includes Sen [27], Zwicker[30], and Saari [23]). But, the mathematical complexity of this analysis has restricted most results to only three candidates.

Here I prove for any number of candidates that all ranking irregularities (not just pairwise ranking cycles but any non-transitive pairwise outcomes) (Section 6), any tallies which do not satisfy an extremely strong consistency property which I call *additive transitivity* (Section 5.2), any difference between the BC and pairwise tallies (Section 7.2), all differences in BC rankings over different subsets of candidates, all differences between methods using pairwise rankings or tallies (Section 7.2), etc., etc., are completely caused by what I call (when introduced in Section 6) the *Condorcet differentials.* In other words, by understanding the Condorcet *deviation* terms as developed here, several major mysteries of this research area finally are explained and answered. (As one of several illustrations, recall that a common difficult issue has been to determine the likelihood of non-transitive pairwise outcomes; the answer becomes fairly immediate with this approach.) To simplify this analysis, I introduce a new tool that I call *fundamental cycles* (Section 6.3); these cycles help capture much of the just described effects. To explain the meaning of cycles and why procedures have differing outcomes, I discuss Arrow's Theorem (Section 8) in terms of the decomposition.

2.2 Geometry

"Geometry" is my standard tool. For instance, a common difficulty of this research area is to describe profiles in a way which assists in the analysis of procedures. To address and provide an new approach for this problem, I use a "geometric representation of profiles" (Section 4) which allows us to quickly tally election outcomes and which provides insight into why rankings can change with the number of candidates.

Geometry also provides intuition about the conflicting roles of the Basic and Condorcet profiles. For instance, in Sections 5.4, 7.1 all pairwise outcomes are identified in terms of the different roles of the Basic and Condorcet profiles. This is used, for instance, to explain (e.g., Section 7.2.1) why cycles and other non-transitive behavior dominate as the number of alternatives grows.

A main goal of this paper is to use these new structures to extract new results about the BC, Condorcet winner, and an assortment of other procedures which rely upon pairwise votes (e.g., Section 7). With geometry, I also characterize all profiles and ways the BC and pairwise rankings can differ. Among new results, I generalize the known assertion that the BC ranks the Condorcet winner (a candidate who beats all other candidates in pairwise elections) strictly above a Condorcet loser (a candidate who loses to everyone in pairwise elections). The first known (to me) converse results also are given here; e.g., I prove that if the pairwise election rankings are not "too" intransitive, they must rank the BC winner strictly over the BC loser (Section 7.2). Most proofs are in Section 10.

2.3 Two useful conclusions

I need the following two conclusions.

Theorem 1. *(Saari [16]) Assume there are* $n \geq 3$ *candidates. Select, in any manner, a transitive ranking for each of the* $2^n - (n+1)$ *subsets of candidates. (So the rankings of the different subsets need not be related in any manner.) There* *exists a profile so that the sincere plurality (or pairwise) ranking for each subset is as selected.*

An extension of Theorem 1 to all positional methods (Saari [16, 17]) suggests the likelihood of highly chaotic outcomes (Saari [21]). For technical reasons, I need the regularity ensured by the following Proposition 1. This assertion, which is not obvious (and proved in the companion paper), states that all voting paradoxes are based on a linear combination of the profile's pairwise and plurality tallies over certain subsets of candidates. To describe this result, label in any desired manner all subsets of candidates as

$$S_1,\ S_2,\ \dots,\ S_{2^n-(n+1)}$$

and let $Plur_s(\mathbf{p}, |S_j|)$ be the plurality tally of profile $\mathbf{p}$ for candidate $c_s \in S_j$.

Proposition 1. *For $n \geq 3$ candidates and a specified positional method for set S_k, there are constants $a^{|S_j|}$ so that the method's election tally of any candidate $c_s \in S_k$ for any profile $\mathbf{p}$ is*

$$\sum_{S_j \subset S_k} a^{|S_j|} Plur_s(\mathbf{p}, |S_j|). \tag{2.2}$$

3 Profiles

As a profile specifies the number of voters with each ranking of the candidates, a n-candidate profile can be represented as a point in $R^{n!}$ with non-negative integer entries. To do this, list, in some manner, all $n!$ rankings of the candidates. The jth voter type, then, is the jth ranking. To illustrate by using the following listing for $n = 3$ (see Saari [14, 15]),

Type	Ranking	Type	Ranking
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

(3.1)

$(0, 6, 0, 4, 5, 0)$ is the profile with six type-two, four type-four, and five type-five voters.

Normalized profiles specify the fraction of all voters with each voter type. This space is identified with the $n! - 1$ dimensional simplex

$$Si(n!) = \left\{ \mathbf{x} = (x_1, \dots, x_{n!}) \in R^{n!} \mid \sum_{j=1}^{n!} x_j = 1,\ x_j \geq 0 \right\} \tag{3.2}$$

(So, $(0, 6, 0, 4, 5, 0)$ normalizes to $(0, \frac{6}{15}, 0, \frac{4}{15}, \frac{5}{15}, 0)$.) As normalized profiles can be identified with probabilities, the arguments developed here extend to the literature on individual choice behavior and probabilistic voting. (See the important book (Luce [9]).)

The analysis is further simplified by using the following "profile differentials."

Definition 1. *An* integer profile differential *is the difference between two profiles which involve the same number of voters. A* normalized profile differential *is the difference between two vectors from the normalized profile space $Si(n!)$.*

The sum of a profile differential's entries equals zero. Although a differential allows *negative* numbers of voters, this causes no problems in computing election outcomes.

3.1 Universal kernel

To convert a profile differentials into a profile, notice that profile $\mathbf{K}^n$, defined to have one voter for each of the $n!$ types, has completely tied outcomes for all positional procedures and subsets of candidates. By adding scalar multiples of $\mathbf{K}^n$ to a profile differential, a resulting profile has the same election rankings and differences in tallies as the differential. To illustrate, as -3 is the most negative value in the differential $(-3, 1, 1, -2, 3, 0)$, add $3\mathbf{K}^3$ to obtain the profile $(0, 4, 4, 1, 6, 3)$ which has the same election rankings as the differential.

The next result states that many profiles other than $\mathbf{K}^n$ have no effect on election rankings.

Theorem 2. *For $n \geq 3$, there exists a $n! - 2^{n-1}(n-2) - 2$ dimensional subspace of profile space $Si(n!)$, called* the universal kernel *and denoted by $\mathcal{UK}^n$, so that if $\mathbf{p} \in \mathcal{UK}^n$, then all positional and pairwise election outcomes of any set of candidates is a complete tie.*

For $n = 3$, $\mathcal{UK}^3$ has dimension zero; it is the barycentric point $\frac{1}{6}\mathbf{K}^3$. Consequently, any three-candidate profile with neutral election outcomes must be a scalar multiple of $\mathbf{K}^3$. A surprise is the rapid growth of the $\mathcal{UK}^n$ dimension with n; e.g., with $n = 4$ candidates, it is six-dimensional and for $n = 5$ it is 70-dimensional. Indeed, as asserted next, those profiles with no effect on election rankings quickly dominate profile space.

Corollary 1. *For $n \geq 5$ candidates, the dimension of $\mathcal{UK}^n$ is more than half the dimension of profile space. Namely, the dimension of the subspace of profiles which lead to a complete tie in all positional elections over all subsets of candidates exceeds the dimension of the profile subspace that causes any difference in some election outcome.*

The ratio of the $\mathcal{UK}^n$ and $Si(n!)$ dimensions rapidly approaches unity as $n \to \infty$.

Proof. According to Theorem 2, the second assertion follows by finding the limit of

$$\frac{n! - 2^{n-1}(n-2) - 2}{n!} = 1 - \frac{2^{n-1}(n-2) - 2}{n!} \tag{3.3}$$

which rapidly approaches unity as $n \to \infty$. The first assertion requires finding where the Eq. 3.3 ratio is greater than one half. A direct computation proves this is true for $n \geq 5$. ☐

Dramatic illustrations of Cor. 1 are that $n = 6$ and $n = 7$ give, respectively, Eq. 3.3 ratios of 0.8194 and 0.9361. With just seven candidates, then, $\mathcal{UK}^7$ consumes over 93% of the $(7! - 1) = 5039$ dimensions of $Si(7!)$. This suggests that one reason choice theory has proved to be so complicated is that effort is spend analyzing profiles which, in fact, have no impact on election rankings. A significant simplification in the analysis, then, occurs by concentrating only on the *effective profiles* which actually determine election outcomes.

3.2 Kernel profiles

While it is important to characterize $\mathcal{UK}^n$, the large dimensions of $\mathcal{UK}^n$ prove that a full analysis would dominate the discussion. Thus, I provide a complete description of $\mathcal{UK}^n$ for $n = 4$ and a nearly complete one for $n \geq 5$.

To create $\mathcal{UK}^n$ entries, take the difference between two profiles with identical plurality tallies for each subset of candidates. To illustrate, start with a profile differential where 1 voter has the ranking $(C \succ D) \succ (A \succ B)$ and -1 have $(C \succ D) \succ (B \succ A)$. The cancelling effect of the -1 term forces all plurality tallies in the four candidate subset and the four triplets to be zero. (In these elections, only C or D is top-ranked.) Only the $A \succ B$ pairwise outcome (tally 1:-1) avoids a tie. These tallies determine (Proposition 1) all positional tallies over all subsets of candidates.

Identical tallies occur by replacing the $(C \succ D)$ portion of both rankings with $(D \succ C)$. The difference between these profiles defines the $\mathcal{UK}^4$ profile differential

Number	Ranking	Number	Ranking
1	$C \succ D \succ A \succ B$	-1	$C \succ D \succ B \succ A$
-1	$D \succ C \succ A \succ B$	1	$D \succ C \succ B \succ A$

(3.4)

By changing the identity of the candidates in each pair, we obtain $\binom{4}{2} = 6$ versions of Eq. 3.4.

To extend this argument to $n \geq 4$ candidates, partition the n candidates into two sets G_1, G_2, where each G_i has at least two candidates. Let r_i be a strict ranking of the G_i candidates, $i = 1, 2$. Let $\sigma(r)$ be a permutation of the r ranking, and let $\rho(r)$ be the special permutation which reverses the ranking r. (So, $\rho(A \succ B \succ C \succ D) = D \succ C \succ B \succ A$; one σ choice is $\sigma(A \succ B \succ C \succ D) = D \succ A \succ B \succ C$.)

Choose non-identity permutations σ_j for the candidates in G_j, $j = 1, 2$. The plurality and pairwise tallies of any subset with a G_1 candidate are zero with the profile differential where 1 voter has the preference $r_1 \succ r_2$ and -1 have $r_1 \succ \sigma_2(r_2)$. The plurality and pairwise tallies for a G_2 candidate in a subset without G_1 candidates depends on the choice of σ_2. Whatever these tallies, identical pairwise and plurality tallies arise with the profile differential where 1 voter has $\sigma_1(r_1) \succ r_2$ and -1 have $\sigma_1(r_1) \succ \sigma_2(r_2)$. According to Proposition 1, the difference between these profile differentials has a zero tally for all candidates in all subsets. The difference, the *symmetry changing profile differential*,

Number	Ranking	Number	Ranking
1	$r_1 \succ r_2$	-1	$r_1 \succ \sigma(r_2)$
1	$\sigma_1(r_1) \succ \sigma_2(r_2)$	-1	$\sigma_1(r_1) \succ r_2$

(3.5)

is in $\mathcal{UK}^n$. The special case (and only choice for $n = 4$) where $\sigma_j = \rho$ is called *the double reversal profile differential.*

Theorem 3. *$\mathcal{UK}^4$ is spanned by the six double reversal profile differentials; i.e., all kernel vectors are weighted sums of double reversal profile differentials and $\mathbf{K}^4$. For $n \geq 5$, all symmetry changing profile differentials are in $\mathcal{UK}^n$. All kernel profiles are weighted sums of $\mathbf{K}^n$ and $\mathcal{UK}^n$ profile differentials.*

Proof. It remains to prove that the six vectors described for $n = 4$ are linearly independent. As each entry of each vector involves a $Si(4!)$ component not in any other vector, the conclusion is immediate. ☐

The $\mathcal{UK}^n$ profiles have no effect on election outcomes for any subset, so the smaller (once $n \geq 5$) dimensional orthogonal subspace, *the subspace of effective profiles $\mathcal{EP}^n$*, determines all election outcomes of all subsets of candidates for all positional procedures. Before analyzing these profiles, it is worth noting that the huge $\mathcal{UK}^n$ space is mischief in waiting. After all, any new election procedure which does *not* have tied outcomes for $\mathcal{UK}^n$ profiles generates new paradoxes. (Components which require a completely tied outcome for pairwise and positional methods can have non-tied outcomes for the new procedures.)

4 Representation tetrahedrons and simplices

A standard difficulty is to find profile representations which assist the analysis. One approach (Saari [14, 15]) uses simplices with vertices equal distance from each other. (This equilateral simplex in R^{n-1} is an equilateral triangle for $n = 3$.) Associate each of the $n \geq 2$ candidates $\{c_1, c_2, \dots, c_n\}$ with a vertex; a ranking is assigned to a simplex point according to its distance from each vertex where "closer is better." A simplex divided into "ranking regions" is the "representation simplex."

The $n = 4$ simplex is an equilateral tetrahedron where each face is an equilateral triangle. "Open" this "representation tetrahedron" by cutting along the edges from the D vertex to obtain Figure 1. Each face is defined by three candidates, so the missing candidate on a face corresponds to the tetrahedron vertex most distant from the face; she is bottom ranked. Thus the Figure 1 ranking region with a "•" in the B-C-D face represents $C \succ B \succ D \succ A$.

Represent a profile by listing the number of voters with each ranking in the appropriate ranking region. To compute election tallies, notice that A is top-ranked in those regions with A as a vertex. Thus, the sum of terms in the lightly

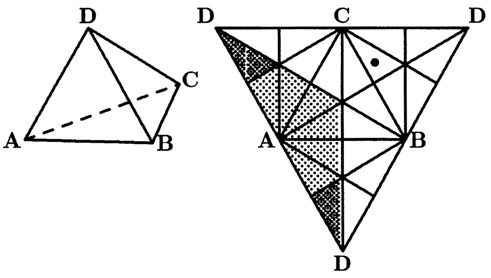

Figure 1. Representation tetrahedron

shaded region of Figure 1 determine A's four-candidate plurality tally. When only candidates $\{A, B, C\}$ are considered, A also receives votes from voters who have A second-ranked and D top-ranked, so A's tally is augmented by the values in the two heavier shaded regions. (A similar description holds for the other three-candidate subsets.) This geometry already explains why rankings can differ as candidates are added or dropped.

In the $\{A, B\}$ pairwise election, A's tally is the sum of numbers to the left of the middle $A \sim B$ line; B's tally is the sum to the right. This line connects the D vertex at the bottom of the figure to the C vertex on the top. A similar description holds for all other pairwise elections which do not involve D. In the $\{A, D\}$ pairwise election, however, the $A \sim D$ line is affected by how the tetrahedron is opened. Here, A's tally is the sum of points in the square where two edges are the $A \sim D$ lines in the $\{A, B, D\}$ and $\{A, C, D\}$ faces. The last edge connects vertices B and C; it separates two faces of the tetrahedron.

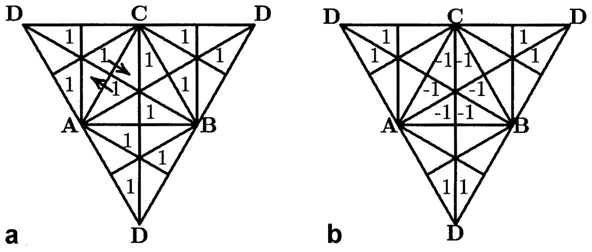

Figure 2a,b. A kernel and a basic profile

Applying these counts to the Figure 2a profile (where each face has a Condorcet triplet; see Eq. 2.1), we find that all plurality and pairwise outcomes are tied. According to Proposition 1, this is a $\mathcal{UK}^4$ profile. The rankings of *all* subsets for the Figure 2b profile agree with $D \succ A \sim B \sim C$.

The arrows in Figure 2a represent a *double reversal profile differential*. The tip of each arrow is a voter's new ranking (so a voter is assigned for each of $(A \succ C) \succ (D \succ B)$, $(C \succ A) \succ (B \succ D)$ while the base represents the former preferences (so -1 voters are assigned for each of $(A \succ C) \succ (B \succ D)$, $(C \succ A) \succ (D \succ B)$). All double reversal profiles have this geometric description; the six edges define the six double reversal profiles; e.g., double reversal differentials symmetrically move voters across an edge to adjacent faces. With this geometric representation, it is easy to show that adding appropriate double reversal differentials to $\mathbf{K}^4$ defines twice the Figure 2a profile.

5 Basic and Borda profiles

The $\mathcal{EP}^n$ *effective profiles* described here determine all pairwise and BC outcomes. The *Basic profiles* are where the tallies of all procedures over all subsets agree; the *Borda profiles* are closely related. The *Condorcet subspace* (introduced in Section 6) is responsible for all cyclic and non-transitive pairwise outcomes, conflict among pairwise rankings, between the pairwise and BC rankings, and paradoxes and difficulties of procedures based on pairwise rankings including those introduced by Copeland [25, 11], Kemeny [8, 26], and Arrow and Raynaud [2].

5.1 Basic profiles

The definition of the Basic profiles is surprisingly simple.

Definition 2. *A n-candidate* Basic profile differential for candidate c_j, *denoted by* $\mathbf{B}^n_{c_j}$, *assigns a voter for each ranking where c_j is top-ranked and -1 voters for each ranking where c_j is bottom-ranked.*

The $\mathbf{B}^4_D$ Basic profile is displayed in Figure 2b; the $\mathbf{B}^4_D$ election outcome for all positional procedures over all subsets of candidates is $D \succ A \sim B \sim C$.

Definition 3. *A normalized positional voting vector for k candidates is*

$$\mathbf{w}^k = (w_1, w_2, \dots, w_{k-1}, w_k), \quad w_1 = 1,\ w_k = 0,\ w_i \geq w_{i+1} \geq 0,\ i = 1, \dots, k-1 \tag{5.1}$$

In tallying a $\mathbf{w}^k$ election, w_j points are assigned to a voter's jth ranked candidate, $j = 1, \dots, k$. The candidates are ranked according to the sums of assigned points.

A normalized BC election for k candidates is where

$$\mathbf{w}^k = \mathbf{b}^k = \left(1, \frac{k-2}{k-1}, \frac{k-3}{k-1}, \dots, 0\right). \tag{5.2}$$

The voting vector for a pair, $(1, 0)$, establishes pairwise voting as a special case of positional voting. The following fundamental theorem asserts that nothing goes wrong with Basic profiles.

Theorem 4. *For each subset of candidates, the $\mathbf{B}^n_{c_j}$ tallies of all normalized positional voting procedures agree. In particular, for $k \geq 2$, if c_j is in a k-candidate subset of candidates, then for* any $\mathbf{w}^k$, *the respective tally for c_j and for any other candidate is*

$$(n-1)!\ \textit{points}, \quad -\frac{(n-1)!}{k-1}\ \textit{points}. \tag{5.3}$$

If c_j is not in a k-candidate subset, then all candidates receive zero points.

The Basic profile differentials define a $(n-1)$-dimensional subspace spanned by any $(n-1)$ of the $\{\mathbf{B}^n_{c_j}\}^n_{j=1}$. Indeed,

$$\sum_{j=1}^{n} \mathbf{B}^n_{c_j} = \mathbf{0} \tag{5.4}$$

Equation 5.4 permits assuming that a Basic profile $\sum_{j=1}^{n} a_j \mathbf{B}^n_{c_j}$ has at least one zero a_j coefficient.

5.2 Power of basic profiles

To appreciate the Basic profiles, recall how much of choice theory is devoted toward understanding why the outcomes of different procedures can differ. As all procedures for all subsets of candidates must agree on the Basic profile, the Basic profiles capture the sought after state of agreement.

Indeed, the tallies of the Basic profile even satisfy a powerful *additive transitivity property*. To explain, the ultimate level of transitivity is attained should the tallies share the additive properties of points $x_1, x_2, \ldots, x_k$ on the line given by

$$(x_1 - x_2) + (x_2 - x_3) + \ldots + (x_{k-1} - x_k) = x_1 - x_k. \tag{5.5}$$

Not only is the additive Eq. 5.5 condition normally missing from election tallies, but even the weaker ordinal ranking conditions need not be satisfied. However, the Basic profile tallies of *all* positional procedures over all subsets transcends transitivity to satisfy a *additive transitivity condition*[1] which closely resembles Eq. 5.5. The tallies mimic Eq. 5.5 even to the extent that the pairwise differences can come from different subsets of candidates and tallying procedures; this, of course, eliminates the severe complexities that have motivated but frustrated choice theory.

Definition 4. *Let S be a set of $|S| = k$ candidates. For profile $\mathbf{p}$, let $\tau^{\mathbf{w}^{|S|}}(A, B)$ be the difference between A's and B's $\mathbf{w}^k$ tally of set S. If $|S| = 2$, denote this difference by $\tau^2(A, B)$. A profile satisfies* additive transitivity *if, for any subset $S = \{c_1, c_2, \ldots, c_k\}$ of candidates, any permutation of the indices, and any $\mathbf{w}^{|S|}$, the differences in tallies satisfy*

$$\sum_{j=1}^{s} \tau^{\mathbf{w}^{|S|}}(c_j, c_{j+1}) = \tau^{\mathbf{w}^{|S|}}(c_1, c_{s+1}). \tag{5.6}$$

With the definition $\mu_k = (n-1)![1 + \frac{1}{(k-1)}]$ I now can describe the remarkable compatibility of election tallies for Basic profiles.

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[1] My thanks to Duncan Luce for suggesting this term to replace my original choice.

Corollary 2. *For a Basic profile* $\sum_{j=1}^{n} a_j \mathbf{B}^n_{c_j}$, *if* $c_i, c_j \in S_h$, *then for any* $\mathbf{w}^{|S_h|}$, *the difference between their positional tallies is*

$$\tau^{\mathbf{w}^{|S_h|}}(c_i, c_j) = (a_i - a_j)\mu_{|S_h|}. \tag{5.7}$$

So, if $c_1, c_2 \in S_1,$, $c_2, c_3 \in S_2, \dots, c_k, c_{k+1} \in S_k$, *and* $c_1, c_{k+1} \in S_{k+1}$, *where* $\mathbf{w}^{|S_j|}$ *is assigned to* S_j *and where the sets* S_j *may, or may not differ with each* j, *then*

$$(a_1 - a_{k+1}) = \frac{\tau^{\mathbf{w}^{|S_{k+1}|}}(c_1, c_{k+1})}{\mu_{|S_{k+1}|}} = \sum_{j=1}^{k} \frac{\tau^{\mathbf{w}^{|S_j|}}(c_j, c_{j+1})}{\mu_{|S_j|}} = \sum_{j=1}^{k}(a_j - a_{j+1}) \tag{5.8}$$

Proof. This is an immediate consequence of Theorem 4. □

As Cor. 2 holds for *any* positional method and any choice of subsets, the actual result is stronger than Eq. 5.6 as the relative tallies of pairs can come from different subsets with different positional methods. To illustrate with $9\mathbf{B}^5_{c_1} + 7\mathbf{B}^5_{c_2} + 2\mathbf{B}^5_{c_3} + \mathbf{B}^5_{c_4}$, the $\{c_1, c_3\}$ relative plurality tally from $\{c_1, c_2, c_3\}$ plus the $\{c_3, c_5\}$ relative $(1, 1, 1, 1, 0)$ tally from the set of all five candidates plus the $\{c_5, c_2\}$ relative $(1, \frac{1}{3}, \frac{1}{4}, 0)$ tally from $\{c_2, c_3, c_4, c_5\}$ determines, say, the $\{c_1, c_2\}$ relative BC tally for $\{c_1, c_2, c_4\}$to be

$$\frac{\tau^{\mathbf{b}^3}(c_1, c_2)}{\mu_3} = \frac{(9-2)\mu_3}{\mu_3} + \frac{(2-0)\mu_5}{\mu_5} + \frac{(0-7)\mu_4}{\mu_4} = (9-7).$$

So, by knowing the pairwise tallies for a Basic profile, *we know the profile's election tallies for all possible procedures with all possible subsets of candidates.* Conversely, just by knowing a Basic profile's plurality or BC outcome for all n candidates, we know what happens for all pairs and all subsets of candidates with any choices of positional methods. This is a remarkable property.

5.3 Borda profiles

The Basic profiles are derived from the *Borda profiles.*

Definition 5. *A Borda profile differential for candidate* c_j *of the* n *candidates, denoted by* $\mathbf{Bor}^n_{c_j}$, *is where* $n + 1 - 2k$ *voters are assigned to each ranking where* c_j *is* k*th ranked,* $k = 1, \dots, n$.

To motivate the *"Borda"* title, recall that the BC assigns $n - j$ points to a voter's jth ranked candidate. An equivalent method (where the sum of points cast is zero) is to assign $2(n - j) - (n - 1) = n + 1 - 2j$ points for a voter's jth ranked alternative, $j = 1, \dots, n$. These weights agree with the number of voters assigned to the $\mathbf{Bor}^n_{c_j}$ rankings.

Theorem 5. *The* $\{c_j, c_s\}$ *pairwise tally for* $\mathbf{Bor}^n_{c_j}$ *is* $((n + 1)!/6 : -(n + 1)!/6)$; *that is,* $(n + 1)!/6$ *points for* c_j *and* $-(n + 1)!/6$ *for* c_s. *All other pairwise tallies end in* $0 : 0$ *ties.*

Proof. It suffices to find c_j's tally for the rankings where $c_j \succ c_s$. There are $(n-1)!$ rankings where c_j is in kth place; there are $(k-1)$ positions where c_s is ranked above c_j. Each such positioning of the two candidates defines $(n-2)!$ rankings. Thus, there are $(n-1)! - (k-1)(n-2)! = (n-k)(n-2)!$ rankings where c_j is in kth position and ranked above c_s. For each ranking, the number of points c_j receives equals the number of voters, $(n+1-2k)$, with that ranking. Thus, c_j's tally is

$$\begin{aligned}\sum_{k=1}^{n-1}(n+1-2k)(n-k)(n-2)! &= (n-2)!\sum_{k=1}^{n-1}(n^2+(1-3k)n+(2k^2-k) \\ &= (n+1)!/6\,.\end{aligned}$$

The tally of a pair not involving c_j is a zero-zero tie because we are using profile differentials and candidates other than c_j are treated symmetrically. $\square$

It is clear from Thms. 4, 5 that for any n and any choice of $\{a_j\}_1^n$ coefficients, the profiles

$$\sum_{j=1}^{n} a_j \mathbf{B}_j^n \text{ and } \frac{6}{n(n+1)}\sum_{j=1}^{n} a_j \mathbf{Bor}_j^n \tag{5.9}$$

have identical pairwise outcomes. Thus, Borda profiles also satisfy the additive transitivity property. But the Borda profiles do *not* satisfy certain Basic profile properties of requiring common positional tallies. For instance, with the Figure 3a representation of $\mathbf{Bor}_A^4$, the $(1,0,0,0)$ and $(1,1,1,0)$ outcome $A \succ B \sim C \sim D$ has a $18 : -6 : -6 : -6$ tally, the $(1,1,0,0)$ tally is $24 : -8 : -8 : -8$, and the $\mathbf{b}^4$ tally is $20 : -20/3 : -20/3 : -20/3$. The rankings are consistent over the subsets, but we lose remarkable Basic tallying properties such as the ones characterized by Cor. 2.

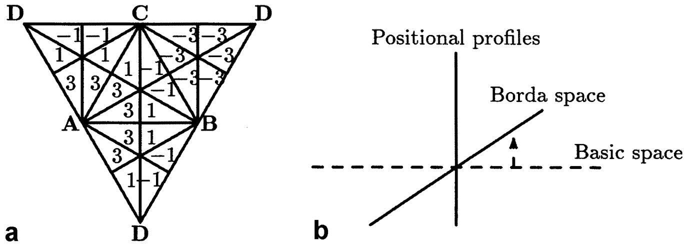

Figure 3a,b. Structure of Borda profiles

Although related by having identical pairwise election outcomes (with the Eq. 5.9 scaling), these profiles differ because (once $n \geq 4$) a Borda profile cannot be represented as a Basic profile and vice versa. Indeed, these two subspaces agree only at the origin (the zero profile). The relative structures of these subspaces is depicted in Figure 3b where one solid line represents the $(n-1)$ dimensional Borda subspace, the other represents profiles which affect positional methods other than the BC. The spaces are not orthogonal as proved by the different $\mathbf{Bor}^4_A$ tallies for different positional election procedures.

A natural approach is to separate the pairwise and positional effects and recover the desired separation of election outcomes by converting the specified subspaces into an orthogonal system. The Basic profiles, represented by the dashed line in the figure, is the result. Of course, the orthogonalization process ensures that there are profile differentials orthogonal to the space spanned by the Basic and Condorcet differentials which have non-zero pairwise tallies; this is illustrated by the Figure 3 arrow. But, these profiles offer nothing new about pairwise outcomes.

So, when emphasizing the precise *profiles* which cause different pairwise outcomes, use the Borda profiles. When considering almost any other issue – such as finding all possible pairwise outcomes, comparing pairwise with positional outcomes, etc. – use the much simpler Basic profiles.

5.4 Representation cube

Not all profiles enjoy the desirable Eq. 5.8 property; e.g., the unanimity profile $A \succ B \succ C$ defines $\tau^2(A,B) = \tau^2(A,C) = \tau^2(B,C) = 1$, so $\tau^2(A,B)+\tau^2(B,C) \neq \tau^2(A,C)$. As this example clearly proves the existence of other profiles which influence the pairwise outcomes, we need to find them. Geometric insight assisting the solution comes from the *n-candidate representation cube* of (Saari [14, 15]). With v voters, define

$$x_{i,j} = \frac{\tau^2(c_i,c_j)}{v}. \tag{5.10}$$

The $x_{i,j}$ values of $1, 0, -1$ mean, respectively, that c_i wins unanimously, ties, and does not receive a single vote in a $\{c_i, c_j\}$ election. Because $x_{i,j} = -x_{j,i}$, the $x_{i,j}$ sign indicates who won the election. Elementary algebra, $x_{i,j}$, and Eq. 5.10 reveal the actual tallies.

By assigning each $x_{i,j}$, $i < j$, to a $R^{\binom{n}{2}}$ axis, a profile defines a point, $\mathbf{q}_n \in R^{\binom{n}{2}}$, of pairwise outcomes. As $x_{i,j} \in [-1,1]$, $\mathbf{q}_n$ is in the $\binom{n}{2}$ fold product of the intervals $[-1,1]$ called the *orthogonal cube.* To find all possible pairwise outcomes, notice that the $x_{i,j}$ values for an unanimity profile are ± 1. Thus, each unanimity profile defines an *unanimity vertex* of the orthogonal cube. The *convex hull* of the unanimity vertices is the *representation cube* $\mathcal{RC}(n)$. Namely, if $\mathbf{V}_i$ is the unanimity vertex for the ith ranking of the candidates, then

$$\mathcal{RC}(n) = \left\{ \mathbf{q}_n = \sum_{i=1}^{n!} \lambda_i \mathbf{V}_i \mid \lambda_i \geq 0, \sum_{i=1}^{n!} \lambda_i = 1 \right\}. \tag{5.11}$$

To see the significance of $\mathcal{RC}(n)$, note that if $\mathbf{E}_i$ is the unanimity profile for the ith ranking, then $\mathbf{V}_i$ is the pairwise tally. A normalized profile is the convex sum $\mathbf{p}_n = \sum_{i=1}^{n!} \lambda_i \mathbf{E}_i$, so the corresponding $\mathcal{RC}(n)$ point is $\mathbf{q}_n = \sum_{i=1}^{n!} \lambda_i \mathbf{V}_i$. This leads to the following result.

Proposition 2. *(Saari [15]) All pairwise election outcomes, for any profile, are characterized by the (rational) points of $\mathcal{RC}(n)$. Namely, the pairwise outcomes for any profile define a point in $\mathcal{RC}(n)$; all (rational) points of $\mathcal{RC}(n)$ define the pairwise election outcomes for some profile. $\mathcal{RC}(n)$ meets all orthants, coordinate planes, coordinate axes, etc. of $R^{\binom{n}{2}}$, so all $3^{\binom{n}{2}}$ rankings of the $\binom{n}{2}$ pairs of candidates are supported by a profile.*

The first part of Proposition 2 identifies all possible pairwise election outcomes as the rational points of $\mathcal{RC}(n)$. The second part means that any listing of pairwise rankings is supported by a profile. If the rankings are not transitive, they are caused by the non-Basic profile differentials. To provide a quick proof of this result, notice that each face of the orthogonal cube is determined by a particular candidate from a particular pair who wins with an unanimous vote; i.e., $x_{i,j} = 1$. On this $x_{i,j} = 1$ face are the two unanimity vertices representing the rankings $r \succ (c_i \succ c_j)$ and $(c_i \succ c_j) \succ \rho(r)$ where r is any fixed strict, transitive ranking of the remaining $(n-2)$ candidates. Now consider a two-voter profile which consists of these two rankings. The $\{c_i, c_j\}$ outcome has c_i unanimously beating c_j. For each remaining pair, one voter has one ranking while the other voter has the reversed ranking, so the outcome for this pair is a tie. Thus, this profile defines the point $x_{i,j} = 1$ on the $x_{i,j}$ axis. As this is true for each axis, the convex combination of these points also is in the representation cube. But this convex set clearly meets all all orthants, coordinate planes, coordinate axes, etc. of $R^{\binom{n}{2}}$, so the assertion holds.

5.5 Transitivity plane

To describe the Basic and Borda profiles we need the following.

Definition 6. *The* transitivity plane *of $\mathcal{RC}(n)$ is the $(n-1)$ dimensional plane passing through the origin of $\mathcal{RC}(n)$ spanned by $\{\mathbf{T}^n_{c_i}\}_{i=1}^n$ where vector $\mathbf{T}^n_{c_i}$ has $x_{i,j} = 1$ for all j (so, c_i unanimously beats each of the other candidates), while $x_{k,j} = 0$ when $j, k \neq i$ (representing a tie vote for each remaining pair of candidates).*

Theorem 6. *The pairwise outcomes of a Basic or a Borda profile are in the transitivity plane of $\mathcal{RC}(n)$. All rankings in the transitivity plane satisfy ordinal and additive transitivity. No unanimity vertex outcome is on the transitivity plane.*

Proof. The pairwise election outcomes for $\mathbf{B}^n_{c_j}$ (or $\mathbf{Bor}^n_{c_j}$) has c_j beating all other candidates with a unanimous vote; all other pairs end in a tie. So $\mathbf{B}^n_{c_j}$ (or $\mathbf{Bor}^n_{c_j}$) defines a *direction* consistent with $\mathbf{T}^n_{c_j}$. The conclusion follows by linearity. The rest of the proof is in Section 10. □

To appreciate the transitivity plane, I review definitions (or properties equivalent to the definition) of procedures defined by the pairwise vote. (It has been known since Borda [3] and Nanson [12] that the BC ranking is the sum of points a candidate receives over all pairwise elections. The non-traditional conditions for Kemeny's rule are in (Saari and Merlin [26]).)

Definition 7. *The BC ranking is equivalent to ranking the candidates according to assigned scores where the jth candidate's score is*

$$BC(c_j) = \sum_{k\neq j} x_{j,k}. \tag{5.12}$$

The Copeland ranking is determined by modifying Eq. 5.12: If $x_{i,j} \neq 0$, replace it with the sign of $x_{i,j}$. (Namely, in a $\{c_i, c_j\}$ pairwise election, assign one point to the winner and -1 points to the loser.) The Copeland outcome is determined by ranking candidates according to the sum of assigned points.

Candidate c_j is the Condorcet winner if she wins all pairwise elections. (That is, if $x_{j,k} > 0$ for all $k \neq j$.) She is the Condorcet loser if she loses all pairwise elections.

A Black-winner is the Condorcet winner when defined. Otherwise, it is the BC winner.

The Kemeny ranking for profile $\mathbf{p}$ is a transitive ranking obtained from $\mathbf{q}_n$ by reversing $x_{i,j}$ values so that

1. *the resulting ranking is transitive, and*
2. *this is done by minimizing the sum of the magnitudes of the reversed terms.*

Equivalently, the Kemeny ranking is the nearest (l_1 distance) transitive ranking region to $\mathbf{q}_n$.[2]

While these procedures can have conflicting election outcomes, no conflict occurs with Basic or Borda profiles. Thus, all differences are due to pairwise outcomes off of this plane.

Theorem 7. *All of the procedures defined in Def. 7 have the same election outcome when the profile is a Basic or a Borda profile.*

Proof. The election ranking for a Basic (Borda) profile is transitive, so the Kemeny, pairwise, Condorcet, Black, and Copeland rankings agree. The BC outcome agrees because of Cor. 2. □

6 Condorcet differentials

The transitivity plane is $(n-1)$-dimensional subspace within the $\binom{n}{2}$-dimensional space $\mathcal{RC}(n)$. So, for $n \geq 4$, the profile deviation differentials that differ from Basic (or Borda) profiles and influence pairwise outcomes define a larger dimensional $\mathcal{RC}(n)$ subspace.[3] The significance of this statement is that (Theorem 7)

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[2] The l_1 distance is the sum of the absolute values of each component.

[3] As this $\mathcal{RC}(n)$ subspace has dimension $\binom{n}{2} - (n-1) = \frac{1}{2}n(n-1) - (n-1) = \frac{1}{2}(n-1)(n-2) = \binom{n-1}{2}$, it is at least equal to that of the transitivity plane iff $\binom{n-1}{2} \geq n-1$ iff $n \geq 4$.

these deviation profiles, which cause all conflict among the recognized procedures, are dominant. Consequently, pairwise difficulties are easy to find for $n = 3$ and must be expected for $n \geq 4$.

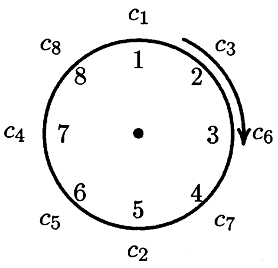

Figure 4. Ranking disk and Condorcet $r = c_1 \succ c_3 \succ c_6 \succ c_7 \succ c_2 \succ c_5 \succ c_4 \succ c_8$ cycle

The building blocks for these profile deviation differentials are the *Condorcet n-tuples* defined, as follows, with the *ranking disk* of Figure 4. Attach to a fixed background a disk that freely rotates about its center. Equally spaced along its circular boundary place the ranking numbers $1, 2, \ldots, n$. To represent a ranking r, place each candidate's name on the fixed background next to the appropriate ranking number. This is the first ranking. Rotate the disk clockwise until number 1 points to the next candidate; this new position defines a second ranking. Continue until n rankings are defined. These n rankings define *the Condorcet n-tuple* denoted by $\mathbf{R}^n_r$.

Notice how the Condorcet triplet (Eq. 2.1) is the Condorcet three-tuple defined by $r = A \succ B \succ C$. Also notice that a Condorcet n-tuple $\mathbf{R}^n_r$ is defined by r, or by any of the n rankings from $\mathbf{R}^n_r$. In other words, the Condorcet n-tuples partition the set of all strict, transitive rankings. (Readers familiar with group theory will recognize $\mathbf{R}^n_r$ as the Z_n orbit of r.)

It is arguable that the election outcome for a Condorcet n-tuple should be a complete tie because, by construction, each candidate is ranked in each position precisely once. But, instead, the outcome for each pair depends upon the relative position of the two candidates in r. For instance, the ranking disk shows that if c_1 is ranked immediately above c_2 in r, then only one $\mathbf{R}^n_r$ ranking has c_2 and c_1, respectively, top and bottom ranked; the remaining $n-1$ rankings have c_1 ranked above c_2. Thus, c_1 beats c_2 with a $n-1:1$ tally. Similarly, if a ranking of the n-tuple has, say, c_4 ranked $s \leq \frac{n}{2}$ candidates above c_1, then c_4 beats c_1 in the pairwise vote with a $n-s:s$ tally. The symmetry of $\mathbf{R}^n_r$ ensures that if c_i is ranked s candidates above a specified candidate in a $\mathbf{R}^n_r$ term, then she is ranked s candidates below another candidate in another $\mathbf{R}^n_r$ term. This proves the following.

Proposition 3. *In $\mathbf{R}^n_r$, a Condorcet n-tuple defined by r, the sum of each candidate's tallies over the $n-1$ pairwise elections all agree. If c_i is ranked s candidates above c_j in a $\mathbf{R}^n_r$ ranking, then the $\{c_i, c_j\}$ election tally is $n-s:s$.*

The following profile differential (an orbit of the group generated by Z_n and a reversal) uses the reversal ranking $\rho(r)$ from Theorem 3.

Definition 8. *The* Condorcet profile differential defined by *r and denoted by $\mathbf{C}^n_r$, assigns one voter for each ranking in the Condorcet n-tuple defined by r and -1 voters for each ranking in the Condorcet n-tuple defined by $\rho(r)$. Namely, $\mathbf{C}^n_r = \mathbf{R}^n_r - \mathbf{R}^n_{\rho(r)}$. A* Condorcet profile differential *is a linear combination of $\mathbf{C}^n_r$ differentials.*

6.1 Condorcet differential tallies

The next statement describes the importance of the Condorcet differentials; they include the subspace of profile deviations needed to understand pairwise voting. Most of the rest of this paper is devoted to understanding the consequences of Thms. 4 and 8.

Theorem 8. *Properties of the Condorcet profile differentials follow:*

1. *If c_i is ranked s candidates above c_j in any $\mathbf{R}^n_r$ ranking, then the $\{c_i, c_j\}$ election tally of $\mathbf{C}^n_r$ is $n - 2s : 2s - n$. Consequently, the sum of a candidate's pairwise tallies over all possible opponents is zero.*
2. *If $\mathbf{p}$ is a profile differential orthogonal to the space spanned by the Borda and the Condorcet profile differentials, then all pairwise tallies of $\mathbf{p}$ are zero. Consequently, all admissible pairwise election tallies are obtained with the weighted sum of Borda (or Basic) and Condorcet profile differentials; all remaining profile components have no effect upon pairwise or BC outcomes.*
3. *For any n-candidate positional method $\mathbf{w}^n$, the $\mathbf{w}^n$ tally of $\mathbf{C}^n_r$ assigns zero to each candidate.*
4. *The Condorcet profile differential is orthogonal to all Basic, Borda profiles, and double reversal profile differentials.*
5. *The Condorcet profile differentials span a space of dimension $\frac{1}{2}(n-1)!$; this is the Condorcet subspace of deviation profiles.*
6. *Let $r = c_1 \succ c_2 \succ \dots \succ c_n$ and r_1 be the ranking obtained by dropping c_j. In the $(n-1)$-candidate subset defined by dropping c_j, the differential $\mathbf{C}^n_r$ becomes $\mathbf{C}^{(n-1)}_{r_1}$ plus a profile differential where one voter has the ranking $c_{j+1} \succ c_{j+2} \succ \dots \succ c_n \succ c_1 \succ \dots \succ c_{j-1}$ and -1 voters have the reversal of this ranking. This last profile differential has a positive component in the $\mathbf{B}^{n-1}_{c_{j+1}}$ direction.*

According to this theorem, the *Condorcet subspace,* which is generated by all Condorcet deviation profiles, has dimension $\frac{1}{2}(n-1)!$. As advertised, profiles from this subspace determine all possible oddities associated with pairwise voting.

6.2 Interpretation

With our goal of understanding voting procedures, the good news (part 2) is that all concerns about pairwise rankings can be completely analyzed with the Basic (or Borda) and Condorcet profile differentials; all other profile differentials are superfluous as they do not effect the pairwise outcomes. (They affect the tally, but not the differences between tallies and the rankings.) According to part 4, the role of the Condorcet profile differentials differs significantly from the Basic or Borda profile as part 1 requires the Condorcet portion to create a cyclic effect.

More good news (part 3) is that the Condorcet differentials do not affect the positional tallies of all n candidates. An important corollary is the observation that *the only linkage between rankings of pairs and the n-candidate positional procedures comes from the Basic profile portion.* Part 6 explains the mystery why election rankings can change when candidates are dropped; e.g., part 3 requires ties for all n-candidate positional methods with a Condorcet differential. By dropping candidate c_j, the Condorcet differential defines a *new* Condorcet differential (causing ties for all $(n-1)$-candidate positional methods) *plus* a term in the $(n-1)$ candidate Basic direction which does affect all positional tallies. This is the sole source of this pathological behavior for many procedures including those identified next.

Corollary 3. *Suppose a procedure which uses pairwise ranking or tallies to determine its outcome has the property that when used with a Basic profile, its ranking of the candidates agrees with the pairwise ranking of the Basic profile. Any changes in the procedure's rankings caused by dropping candidates is strictly due to the Condorcet portion of a profile. This assertion hold for the BC, Copeland's Method, Condorcet winner, Black's method, and Kemeny's Rule.*

According to Cor. 3, all examples illustrating changes in the BC rankings when candidates are dropped, and most examples illustrating related results for any procedure (with runoffs, etc.) must have a strong Condorcet component. (This is illustrated in Section 7.) Thus, to understand these procedures, we need ways to simplify the study of the Condorcet profiles.

6.3 Fundamental cycles

When $n = 3, 4$, the Condorcet profile differentials are orthogonal to the kernel. This is not true for $n \geq 5$.

Proposition 4. *For $n \geq 5$, there exist symmetry reversal profile differentials from $\mathcal{UK}^n$ which are not orthogonal to $\mathbf{C}^n_r$.*

Proof. For $n \geq 5$, $\mathcal{UK}^n$ includes the symmetry reversal profiles where for any permutations σ_1, σ_2 and $r' = r_1 \succ r_2$, one voter is assigned to each of $r_1 \succ r_2$, $\sigma_1(r_1) \succ \sigma_2(r_2)$ while -1 voters are assigned to each of $r_1 \succ \sigma(r_2)$, $\sigma_1(r_1) \succ$

r_2. Let r and r' be defined by the parentheses of $(c_1 \succ c_2 \succ \ldots \succ c_{n-2}) \succ (c_{n-1} \succ c_n)$. By choosing $\sigma_1(c_1 \succ c_2 \succ \ldots \succ c_{n-2}) = c_2 \succ c_3 \ldots \succ c_{n-2} \succ c_1$ (from $\mathbf{R}^{n-2}_{r_1}$) and $\sigma_2 = \rho$, only ranking r appears in both $\mathbf{C}^n_r$ and the kernel differential. □

So, once $n \geq 5$, the Condorcet subspace includes portions of $\mathcal{UK}^n$. While bases exist which avoid kernel components, the cost of using them is that the differentials obscure natural voting symmetries. Indeed, each $\mathbf{C}^n_r$ defines *fundamental classes of cycles.* To illustrate with $r = c_1 \succ c_2 \succ \ldots \succ c_9$, accompanying the $c_1 \succ c_2, c_2 \succ c_3, \ldots, c_8 \succ c_9, c_9 \succ c_1$ cycle is the cycle involving every other candidate where $c_1 \succ c_3, c_3 \succ c_5, c_5 \succ c_7, c_7 \succ c_9, c_9 \succ c_2, c_2 \succ c_4, c_4 \succ c_6, c_6 \succ c_8, c_8 \succ c_1$, three cycles involving every third candidate to obtain $c_1 \succ c_4, c_4 \succ c_7, c_7 \succ c_1$ with similar cycles starting with c_2 and with c_3. The final cycle uses every fourth candidate to derive $c_1 \succ c_5, c_5 \succ c_9, c_9 \succ c_3, \ldots$. The cycles obtained by skipping more candidates have smaller tally differences. To describe the general behavior, and to introduce the *fundamental cycles*, let the c_j subscript represent j if $j \leq n$ and $j - n$ if $j > n$.

Corollary 4. *The Condorcet profile differential defined by $r = c_1 \succ c_2 \succ \ldots \succ c_n$ defines the primary cycle, where $c_j \succ c_{j+1}$, the sth level cycle, $1 < s < \frac{n}{2}$, where $c_j \succ c_{j+s}$, and (for even values of n) $n/2$ degenerate cycles where $s = n/2$ and $c_j \sim c_{j+s}$, $j = 1, \ldots, n$. If $\frac{n}{s}$ equals integer α, then there are s different s level cycles with α candidates. If $\frac{n}{s}$ is not an integer, then a unique sth level cycle involves all candidates. If $x_{i,j} \neq 0$, then either c_i or c_j immediately precedes the other candidate in precisely one of these primary or secondary cycles. Each of these cycles is called a fundamental cycle. Each pair $x_{i,j}$ is in precisely one fundamental cycle.*

By choosing pairwise rankings from different fundamental cycles, we can construct all possible cycles. The above example, for instance, admits the cycle $c_1 \succ c_3 \succ c_4 \succ c_8 \succ c_1$. As this is *not* a fundamental cycle, the tallies between successive terms vary as specified by Theorem 8. In this way, the fundamental cycles serve as the building blocks for all possible cycles.

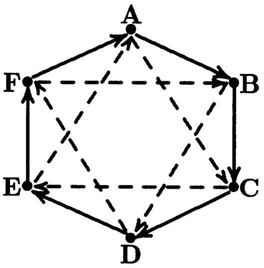

Figure 5. Cyclic arrangements for $\mathbf{C}^6_r$ for $r = A \succ B \succ C \succ D \succ E \succ F$

Figure 5 illustrates how to find all $\mathbf{C}^6_r$ cycles, $r = A \succ B \succ \ldots \succ F$. Place the candidates along a ranking disk according to the choice of r. Next, connect adjacent candidates with an arrow where $A \to B$ means $A \succ B$; this defines the primary cycle. To define the sth level cycle, connect every sth candidate. In Figure 5, then, there is a first-level cycle, and two second-level cycles. The $s = 3$ degenerate cycles are not depicted; they are lines connecting opposite candidates with zero tallies. All $\mathbf{C}_r^6$ cycles are obtained by following the arrows; e.g., $A \succ C \succ E \succ F \succ A$. For all $\mathbf{C}_r^n$, all cycles and pairwise comparisons can be extracted from such figures.

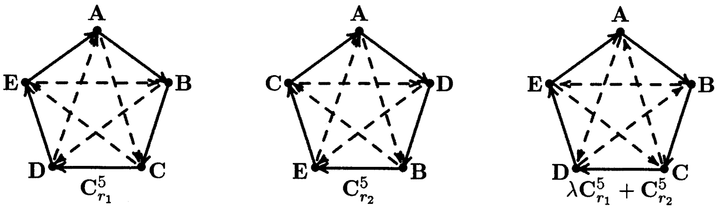

Figure 6. Combining differentials where $1 < \lambda < 3$

A similar approach (illustrated by Figure 6) shows how to compute the cycles associated with sums of profile differentials. As $\lambda > 1$ in the Figure 6 example, the pairwise tallies from the $\mathbf{C}_{r_1}^5$ primary cycle dominate the tallies for the same pairs from $\mathbf{C}_{r_2}^5$; this explains the outside arrows on the third diagram. The λ upper bound requires differences from pairs in $\mathbf{C}_{r_2}^5$'s primary cycle to dominate tallies from $\mathbf{C}_{r_1}^5$'s secondary cycle. Because of the r_2 choice (the reversal of a ranking defined by the dashed lines in the first figure), all remaining rankings differ from $\mathbf{C}_{r_1}^5$. Again, all possible cycles generated by the sum are determined by following the arrows in the diagram. Similarly with all $\{\mathbf{C}_r^n\}$ differentials, all possible pairwise cycles and non-transitive behavior are created.

7 Effects of pairwise votes

According to Theorem 8, only the Basic (or Borda) and Condorcet portions of a profile affect pairwise votes. The following describes useful relationships.

Theorem 9. *For $n \geq 3$, the $\frac{1}{2}(n-1)!$ distinct Condorcet differentials $\{\mathbf{C}_{r_j}^n\}_{j=1}^{(n-1)!/2}$ are pairwise orthogonal. The $\mathcal{RC}(n)$ direction defined by a Condorcet differential is orthogonal to the transitivity plane.*

For $n = 3$, the one-dimensional Condorcet space is spanned by $\mathbf{C}_r^3$ where $r = A \succ B \succ C$. For $n = 4$, the three-dimensional space is spanned by $\{\mathbf{C}_{r_j}^4\}_{j=1}^3$ where $r_1 = A \succ B \succ C \succ D$, $r_2 = A \succ C \succ B \succ D$, and $r_3 = A \succ B \succ D \succ C$. The $\mathcal{RC}(4)$ directions defined by these three differentials are pairwise orthogonal.

For $n \geq 5$, the $\frac{1}{2}(n-1)!$ Condorcet differentials $\{\mathbf{C}_{r_j}^n\}_{j=1}^{(n-1)!/2}$ are pairwise orthogonal, but their $\mathcal{RC}(n)$ directions span a space of dimension $\binom{n-1}{2}$, so they cannot be pairwise orthogonal.

Proof. Each ranking is in precisely one $\mathbf{C}_r^n$ and each $\mathbf{C}_r^n$ is generated (up to $\pm\mathbf{C}_r^n$) by any of the $2n$ rankings in $\mathbf{C}_r^n$. Thus, there are precisely $n!/2n = \frac{1}{2}(n-1)!$

distinct choices of $\mathbf{C}^n_r$. As each ranking is in only one $\mathbf{C}^n_r$, the $\{\mathbf{C}^n_{r_j}\}_{j=1}^{(n-1)!/2}$ differentials are pairwise orthogonal.

The orthogonality of Condorcet tallies with the transitivity plane holds if the tally of any Basic profile $\mathbf{B}^n_{c_k}$ is orthogonal to that of any $\mathbf{C}^n_{r_j}$. The $\mathcal{RC}(n)$ direction $\mathbf{T}^n_{c_k}$ has c_k unanimously beating all candidates; all other directions are zero. Without loss of generality, let $k = 1$ so each $x_{1,j} = 1$. According to Cor. 4, the $\mathbf{C}^n_r$ tallies are described by fundamental cycles. Thus if j is such that $x_{1,j} \neq 0$ for $\mathbf{C}^n_r$, then c_j either immediately precedes or follows c_1 in one fundamental cycle. In either case, there is a unique candidate c_i so that c_1 is between c_i and c_j in the cycle and, according to Cor. 4, $x_{1,j} = -x_{1,i}$. In the scalar product with $\mathbf{T}^n_{c_1}$, terms cancel pairwise according to the fundamental cycles, so the product is zero.

The assertion for $n = 3$ follows from the dimension of the space of Condorcet profile differentials. For $n = 4$, start with any four-candidate ranking, say $r_1 = A \succ B \succ C \succ D$, and compute the associated $\mathbf{C}^4_{r_1}$. Next, choose a ranking which differs from the eight $\mathbf{C}^4_{r_1}$ rankings, say $r_2 = A \succ C \succ B \succ D$, and compute $\mathbf{C}^4_{r_2}$. The last $\mathbf{C}^4_{r_3}$ is determined by one of the remaining eight rankings, say $r_3 = A \succ B \succ D \succ C$.

If $\mathbf{q}_4^{C;j}$ is the $\mathcal{RC}(4)$ direction defined by $\mathbf{C}^4_{r_j}$, $j = 1, 2, 3.$ then

	$x_{A,B}$	$x_{A,C}$	$x_{A,D}$	$x_{B,C}$	$x_{B,D}$	$x_{C,D}$
$\mathbf{q}_4^{C;1}$	1	0	−1	1	0	1
$\mathbf{q}_4^{C;2}$	0	1	−1	−1	1	0
$\mathbf{q}_4^{C;3}$	1	−1	0	0	1	−1

(7.1)

By computing scalar products, it follows that the rows are orthogonal.

The assertion for $n \geq 5$ is an immediate consequence of the dimensions. □

All pairwise outcomes are determined by the Basic and the Condorcet portions, but once $n \geq 5$, the Condorcet portion affect positional outcomes for subsets of candidates. To correct this, a subspace is described in Section 9 which affects only pairwise tallies. For issues involving just pairwise voting, it can be easier to use $\{\mathbf{C}^n_{r_j}\}_{j=1}^{(n-1)!/2}$.

7.1 Geometry of pairwise vote

According to Theorem 9, each $\mathbf{q}_n \in \mathcal{RC}(n)$ has a representation

$$\mathbf{q}_n = \mathbf{q}_n^T + \mathbf{q}_n^C \tag{7.2}$$

where $\mathbf{q}_n^T$ is in the transitivity plane and determined by the Basic (or Borda) profile while $\mathbf{q}_n^C$ is in orthogonal subspace and determined by the Condorcet portion of the profile. As asserted next, this representation is unique.

Proposition 5. *If* $\mathbf{q}_n \in \mathcal{RC}(n)$ *can be represented as*

$$\mathbf{q}_n = \mathbf{q}_n^T + \mathbf{q}_n^C = \mathbf{q}_n^T * + \mathbf{q}_n^C *,$$

then $\mathbf{q}_n^T = \mathbf{q}_n^T *$ *and* $\mathbf{q}_n^C = \mathbf{q}_n^C *$.

Proof. By collecting terms, the differences $\mathbf{q}_n^T - \mathbf{q}_n^T * = \mathbf{q}_n^C * - \mathbf{q}_n^C$ are in orthogonal subspaces, so the assertion holds. □

This structure provides a simple way to understand why procedures have different outcomes. The approach is to understand how each procedure treats the Condorcet portion of a profile. As the first assertion identifies the BC and $\mathbf{q}_n^T$ rankings, it is convenient to compare other procedures with the BC.

Theorem 10. *Let* $\mathbf{q}_n = \mathbf{q}_n^T + \mathbf{q}_n^C \in \mathcal{RC}(n)$ *be an pairwise election outcome.*

1. *The BC outcome and tally are completely determined by the transitivity plane component* $\mathbf{q}_n^T$*; the* $\mathbf{q}_n^C$ *term has no effect upon the BC outcome for n candidates. Thus the BC outcome for* $\mathbf{q}_n$ *assumes the ranking and tallies of the point in the transitivity plane which is closest (in Euclidean distance) to* $\mathbf{q}^n$.
2. *All differences in the pairwise and BC rankings are due to the Condorcet portion of a profile. If the BC outcome is given by* $\mathbf{q}_n^T$*, then all possible ways the BC and pairwise ranking can differ is determined by all possible choices of* $\mathbf{q}_n^C$. *Consequently, all conflict can be designed by adding appropriate Condorcet components to the Basic component of a profile.*
3. *There exist an open set of choices of* $\mathbf{q}_n^T$ *and* $\mathbf{q}_n^C$ *so that the pairwise rankings of* $\mathbf{q}_n^T + \mathbf{q}_n^C$ *are transitive and differ from the BC ranking. More generally, whenever the BC top-ranked (bottom-ranked) candidate is not the Condorcet winner (loser), the difference is due to* $\mathbf{q}_n^C$ *and the Condorcet portion of a profile.*
4. *With no pairwise ties, the Copeland method replaces* $\mathbf{q}_n$ *with the nearest vertex of the orthogonal cube. The Copeland ranking is the ranking of the unique point in the transitivity plane which is closest to the Copeland vertex. All differences between the Copeland and BC outcomes are due to the* $\mathbf{q}_n^C$ *component moving the* $\mathbf{q}_n$ *outcome into a region where the Copeland vertex differs from the vertex of the ranking region with* $\mathbf{q}_n^T$.
5. *Differences between the BC and the Black's method rankings are due to the effects of* $\mathbf{q}_n^C$ *and the Condorcet portion of the profile altering the pairwise rankings.*
6. *Any difference between the Kemeny and BC outcomes; any difference in their properties as candidates are dropped, etc., are due to the Condorcet portion of a profile.*

7.2 Geometry of procedures

The Figure 7 schematic of a representation cube captures aspects of Theorem 10. It follows immediately from additive transitivity that *no* coordinate axis of the representation cube is parallel to the transitivity plane (denoted by the dashed line in the figure). This slanting orientation is the source of many of the complexities of voting procedures.

To use this geometry, notice that the orthogonal projection of $\mathbf{q}_n$ to the transitivity plane defines $\mathbf{q}_n^T$. Thus, the slant of the transitivity plane makes it easy to design settings where $\mathbf{q}_n$ and $\mathbf{q}_n^T$ are in different ranking orthants, so they have different rankings. Consequently, we must expect procedures to differ in outcomes depending on the emphasis they place on the Basic and Condorcet portions. This is illustrated in what follows.

7.2.1 Borda count

As Theorem 8 ensures, the sum of votes a candidate receives in $\mathbf{C}_r^n$ pairwise tallies is zero. As a candidate's BC score is the sum of her tallies over all pairwise comparisons, and as a profile is the sum of the Basic and Condorcet portions, it follows that the BC cancels the Condorcet portion of a profile. Consequently, the BC tally is strictly determined by the Basic (or Borda) portion of a profile.

For a geometric description, we know from the orthogonality properties of Theorem 10 that the transitivity plane component of $\mathbf{q}_n \in \mathcal{RC}(n)$ (the BC outcome) is the closest transitivity plane point to $\mathbf{q}_n$. So, the BC admits the geometric description of using $\mathbf{q}_n$ as the center of, what I call, the *BC sphere.* Expand the sphere's radius (Figure 7) until it first touches the transitivity plane; this point defines the BC ranking and tallies.

As the only (but major) difference between pairwise and BC outcomes is the Condorcet component, we now can geometrically display all examples illustrating all conflict between pairwise and BC outcomes. To illustrate, consider the profile

$$[6\mathbf{B}_A^4 + 5\mathbf{B}_B^4 + 2\mathbf{B}_C^4] + t\mathbf{C}_r^4, \quad r = B \succ A \succ D \succ C \tag{7.3}$$

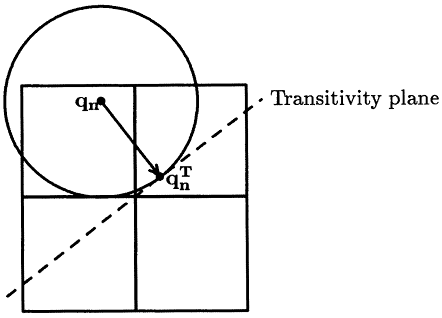

Figure 7. Finding $\mathbf{q}_n$ BC outcome

with Basic profile outcome $A \succ B \succ C \succ D$ and pairwise tallies (recall, for $n = 4$, $\mu_2 = 2(3!)$)

$$\begin{array}{ccc}
\hline
\{A,B\} & \{A,C\} & \{A,D\} \\
\hline
(\mu_2 - 2t, -\mu_2 + 2t) & (4\mu_2, -4\mu_2) & (6\mu_2 + 2t, -6\mu_2 - 2t) \\
\{B,C\} & \{B,D\} & \{C,D\} \\
(3\mu_2 - 2t, -3\mu_2 + 2t) & (5\mu_2, -5\mu_2) & (2\mu_2 - 2t, -2\mu_2 + 2t) \\
\hline
\end{array} \qquad (7.4)$$

Notice that the $A \succ C$ and $B \succ D$ rankings and tallies remain fixed for all t values. All remaining pairwise rankings, however, are eventually reversed with an appropriately strong $\mathbf{C}^4_r$ effect (i.e., with sufficiently large $|t|$ values). The t-bifurcation value for each pair is the t-value with a zero tally. Figure 8 shows these positions (the bullets) along with the five different strict rankings defined by different t values.

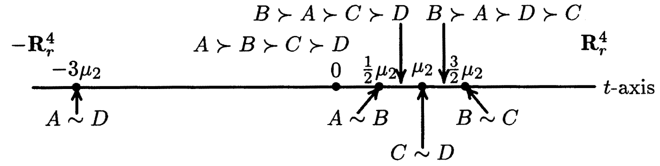

Figure 8. Ranking changes caused by Condorcet term

According to Figure 8, for $-3\mu_2 < t < \frac{1}{2}\mu_2$ the pairwise rankings agree with the BC and Basic ranking. Transitive rankings persist even with stronger Condorcet effects, as manifested by $\frac{1}{2}\mu_2 < t < \mu_2$ and $\mu_2 < t < \frac{3}{2}\mu_2$, but the rankings become, respectively, $B \succ A \succ C \succ D$ and $B \succ A \succ D \succ C$. Notice how the Condorcet winner of both rankings differs from the BC winner and the Condorcet loser for the second ranking differs from the BC loser. Thus this example illustrates the Theorem 10 assertion that all such differences between the pairwise and BC rankings are due to the Condorcet portion. (Illustrating profiles come from Figure 8.) To determine whether the standard of the field, the Condorcet winner, or the BC ranking should be adopted, we must understand the effects of the Condorcet portion as developed in Section 8.

When t satisfies $t < -3\mu_2$ or $t > \frac{3}{2}\mu_2$, the Condorcet portion dominates the Basic portion to force cycles where the sign of t determines which $\pm\mathbf{R}^4_r$ cycle occurs. So, the combination of Figures 7 and 8 and this description provide new connections between BC rankings and other pairwise ranking behavior.

For any n, creating figures such as Figure 8 is immediate by recognizing the central role of the Basic profile $\sum_{j=1}^{n} a_j \mathbf{B}^n_{c_j}$ coefficients. (Without loss of generality, assume that $a_1 \geq a_2 \geq \ldots \geq a_n$.) With a Basic profile, the differences between a_j coefficients is proportional to the differences between c_j tallies. Thus, all tallies can be completely determined with the $(n-1)$ independent variables $\{a_j - a_{j+1}\}_{j=1}^{n-1}$. Changes in a pairwise tally caused by $\mathbf{C}^n_r$ depend on which pairwise rankings are enforced and which are countered by the tallies of the $\mathbf{C}^n_r$ fundamental cycles. To illustrate, as the Eq. 7.3 coefficients satisfy

$$0 < a_A - a_B < a_C - a_D < a_B - a_C, \qquad (7.5)$$

the selected $\mathbf{C}^4_r$, generated by $r = B \succ A \succ D \succ C$, introduces conflict in the $\{A, B\}$, $\{C, D\}$, and $\{B, C\}$ rankings only when $t > 0$, and conflict for $\{A, D\}$ only when $t < 0$.

An interesting feature (due to the tally properties of the fundamental cycles) is how the changes in pairwise rankings are ordered by the Eq. 7.5 ordering of $(a_j - a_{j+1})$ values. Namely, $a_A - a_B$ is the smallest value, so the $\{A, B\}$ ranking is the first to change, then $\{C, D\}$, and finally $\{B, C\}$. But other a_j values define the same Basic ranking with a different ordering of $(a_j - a_{j+1})$ values. Indeed, it is immediate that a_j values can be selected to support any of the $3! = 6$ possible orderings of the positive $(a_j - a_{j+1})$ values. This means that the same BC *ranking* and $\mathbf{C}^4_r$ term offers *six* versions of Figure 8 where each demonstrates new differences between pairwise and BC rankings. For instance, replacing Eq. 7.5 with $a_B - a_C < a_A - a_B < a_C - a_D$ forces the pairwise rankings to change in the order $\{B, C\}$, $\{A, B\}$, $\{C, D\}$; this generates the rankings $A \succ B \succ C \succ D$, $A \succ C \succ B \succ D$, the three cycle $A \succ C, C \succ B, B \succ A$ with Condorcet loser D, and the rankings determined by $\mathbf{R}^4_r$ (with $\{B, C\}$ and $\{A, D\}$ rankings determined by the Basic term).

The Figure 8 r choice has three pairwise changes in one t direction. For an r_2 which changes two pairs in each direction, add $s\mathbf{C}^4_{r_2}$, $r_2 = C \succ A \succ D \succ B$ to the Eq. 7.3 profile. When $t = 0$, the $s > 0$ direction changes $\{A, C\}$ and $\{B, D\}$ rankings and the $s < 0$ direction changes $\{B, C\}$ and $\{A, D\}$. Equation 7.3 requires this ordering for pairwise changes because, for example, $(a_A - a_C) = (a_A - a_B) + (a_B - a_C) < (a_B - a_D) = (a_B - a_C) + (a_C - a_D)$. Thus for $t = 0$, increasing s values change the Basic ranking to a three-cycle $A \succ B, B \succ C, C \succ A$ with Condorcet loser D, and then to the four-cycle of $\mathbf{C}^4_{r_2}$ with the two remaining rankings determined by the Basic term.

The general situation for profile $[a_A\mathbf{B}^4_A + a_B\mathbf{B}^4_B + a_C\mathbf{B}^4_C + a_D\mathbf{B}^4_D] + t\mathbf{C}^4_r + s\mathbf{C}^4_{r_2}$, where the Basic profile coefficients satisfy the Eq. 7.3 ordering, is indicated in Figure 9. Using different $(a_j - a_{j+1})$ values creates (at least) 3! other figures demonstrating how the same $A \succ B \succ C \succ D$ Basic ranking is accompanied with many other rankings. By adding the third Condorcet term, all possibilities are catalogued.

The numbers in Figure 9 label the 20 rankings generated by t, s values. The notation of Table 7.6 lists four candidates for a transitive ranking (e.g., $BADC$ denotes $B \succ A \succ D \succ C$ for region 10), or has a listing beginning and ending with the same candidate for a cycle and followed by the rankings of the remaining pairs (e.g.,$CBADC; CA, BD$ for region 15 represents a four-cycle accompanied with rankings $C \succ A$, $B \succ D$ while the $ABCA \succ D$ of region 12 means that all candidates in the top-three cycle are preferred to D).

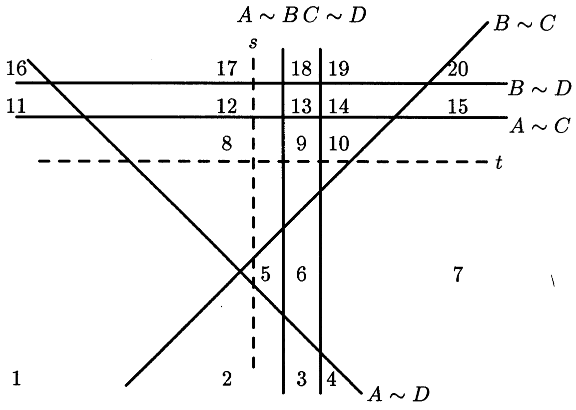

Figure 9. Double Condorcet effects

1	*ABCDA*; *AC*, *BD*	2	*ACBDA*; *AB*, *CD*	3	*ACBDA*; *BA*, *CD*
4	*ACBDA*; *BA*, *DC*	5	*ACBD*	6	$ACBA \succ D$
7	*BADCB* : *BD*, *AC*	8	*ABCD*	9	*BACD*
10	*BADC*	11	*ABCDA*; *CA*, *BD*	12	$ABCA \succ D$
13	*BCAD*	14	$B \succ CADC$	15	*CBADC*; *CA*, *BD*
16	*ABCDA*; *CA*, *DB*	17	*ADBCA*; *AB*, *CD*	18	*ADBCA*; *BA*, *CD*
19	*ADBCA*; *BA*, *DC*	20	*CBADC*; *CA*, *DB*		

(7.6)

Based on the Condorcet and Basic profile structure, the pattern described by Figure 9 and Table 7.6 is to be expected. Namely, the 12 rankings with a four-cycle form an outer circle of regions; here the Condorcet portion of the profile dominates with the larger $|t|$, $|s|$ values.[4] Adjacent to this outer ring are the three regions with a three cycle that either dominates D (regions 6, 12) or is dominated by the fourth candidate (region 14). The interior holds the five transitive ranking regions (5, 8, 9, 10, 13).

To illustrate new results which come immediately from Figure 9 and related figures, consider the rankings generated by $\mathbf{p}_B + u\mathbf{p}_C$ for increasing u values. We might expect that once u forces a non-transitive ranking, the strong Condorcet effect prevents subsequent rankings from being transitive. This is false. Because the union of Figure 9 regions with transitive outcomes is not convex, it is easy to use Figure 9 to select a $\mathbf{p}_C$ direction where the line $u\mathbf{p}_C$ passes from region 8 (with the transitive Basic ranking) to 12 (with non-transitive rankings) to 13 (with a transitive ranking) before entering region 18.

The fact that transitive relationships are destroyed once the Condorcet portion of a profile dominates provides insight into the literature (see Kelly [7])

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[4] As this extends to spatial voting, we should expect new explanations and results. For instance, McKelvey [10] has a nice result asserting that, in general, all candidates are in the top-cycle. The decomposition explains why this should be true because, in general, the effects of the Condorcet terms eventually dominate.

showing that as $n \to \infty$, the likelihood of a pairwise transitive ranking, or of a Condorcet winner and/or loser, etc. tends to zero. Another way to show these varied phenomenon dominate is to compare the ratio of the dimension of the space of Basic profiles to the Condorcet profiles. For this computation, the structure of $\mathcal{RC}^n$ shows that the Basic profiles define a space of dimension $(n-1)$ so a $\binom{n}{2} - (n-1) = \binom{n-1}{2}$ dimensional subspace of the Condorcet subspace causes difficulties. But, different answers are obtained by slightly rephrasing questions. For instance, the space of profiles causing cycles is $\{\text{Kernel}\} \cup \{\text{Condorcet}\}$, and $\dim(\{\text{Kernel}\} \cup \{\text{Condorcet}\})/\dim(Si(n!)) \to 1$ as $n \to \infty$. This comparison is misleading as the dominant dimension comes from the Kernel; indeed, $\dim(\text{Condorcet})/\dim(\text{Kernel}) \to 0, n \to \infty$. A more accurate measure is to determine the $\dim(\text{Basic})/\dim(\text{Condorcet}) = n - 1/\binom{n-1}{2} = 2/n - 2 \to 0$ as $n \to \infty$. On the other hand, $\dim(\text{Condorcet})/\dim(\text{Effective Profiles}) = \binom{n-1}{2}/(2^{n-1}(n-2)+1) \to 0$ as $n \to \infty$, so pairwise voting difficulties constitute only a small portion of all voting problems.

7.2.2 New BC properties and Black's method

To understand Black's method, I explore the relationship between the Condorcet and BC rankings as determined by Basic and Condorcet profile components $\mathbf{p}_B + t\mathbf{p}_C$ where the $\mathbf{p}_C$ portion introduces cyclic effects. As demonstrated by Tables 7.4, 7.6, these cyclic effects are influenced by the coefficients of the Basic profile.

Because $\mathbf{p}_C$ defines rankings associated with the same BC (Basic) ranking, a way to extract BC and Condorcet relationships is to emphasize $\mathbf{p}_C$. The new proof of the following well-known assertion (known at least to Nanson [12]) emphasizes this profile structure.

Corollary 5. *A Condorcet winner (loser) never is BC bottom (top) ranked. If both a Condorcet winner and loser are defined, the Condorcet winner is BC ranked above the Condorcet loser.*

Proof. With profile differentials, each pairwise election for the Condorcet winner has a positive value. The sum of the tallies the Condorcet winner receives in pairwise elections cancels the effects of the Condorcet components, so this sum gives the tallies for the Basic profile component; it gives a positive BC score. Similarly, the sum for a Condorcet loser, which is the candidate's BC score, is negative. The conclusion now follows. ☐

But, what happens if a Condorcet winner or loser does not exist? To answer this question, I use the natural concept of a "layer" which (with different notation) probably has been used by many others.

Definition 9. *From the strict pairwise rankings defined by* $\mathbf{q}_n$, *the* first layer $\mathcal{L}_1$ *is the smallest subset of candidates where each candidate in* $\mathcal{L}_1$ *beats all candidates* *not in $\mathcal{L}_1$. By induction, the jth layer of candidates, $\mathcal{L}_j$, is the smallest subset of candidates where each $\mathcal{L}_j$ candidate is beaten by all candidates in the earlier layers $\mathcal{L}_1, \dots, \mathcal{L}_{j-1}$, but beats all candidates not in $\mathcal{L}_1, \dots, \mathcal{L}_{j-1}, \mathcal{L}_j$.*

It is natural to wonder how the BC ranks candidates from different layers. We might suspect that candidates in a higher ranked layer always are BC ranked above candidates in a lower layer, but even the transitive rankings, where each layer consists of a single candidate, of Table 7.6 proves this is false. A weaker guess is that each candidate in the top layer is BC ranked above each candidate in the bottom layer, but region 14 of Table 7.6 provides counter-examples. Maybe each candidate in the top-layer is ranked above *some* candidate in the bottom layer. To prove this is false, for $n = 6$ let the top layer be defined by c_1, c_2, c_3, and the bottom layer by c_4, c_5, c_6. Let $x_{1,2} = -30t, x_{2,3} = -60t, x_{3,1} = -t$. Let $x_{i,j} = t$ where $i = 1, 2, 3$ and $j = 4, 5, 6$. Let $x_{4,5} = x_{5,6} = x_{6,1} = t$. For a sufficiently small value of $t > 0$, this arrangement defines a point in the representation cube. The Borda scores, however, are $c_1 = [-30 + 1 + 3]t = -26t, c_2 = -27t, c_3 = 62t, c_4 = c_5 = c_6 = -3t$ for the BC ranking of $c_3 \succ [c_4 \sim c_5 \sim c_6] \succ c_1 \succ c_2$ where c_1 and c_2 are ranked below *all* candidates in the bottom cycle.

A remaining possibility is if *some* candidate from the top-layer always is BC strictly ranked above *some* candidate in the bottom layer. This statement is true. To see how it includes Cor. 5 as a special case, notice that if $\mathbf{q}_n$ has a Condorcet winner, then she is the sole occupant of $\mathcal{L}_1$. According to the next theorem, she must be ranked above some candidate in the last layer. Similarly, a Condorcet loser is the only candidate in the last layer. If there is both a Condorcet winner and loser, then there is a single candidate in the top and the bottom layers, so the BC ranks the Condorcet winner above the Condorcet loser.

Theorem 11. *Suppose $\mathbf{q}_n$ defines at least two layers. Some candidate in the top-layer is BC strictly ranked above some candidate in the bottom layer.*

Proof. If c_i is in the top layer, she beats all candidates in lower layers. Thus, $\sum x_{i,j}$, where c_j is *not* in $\mathcal{L}_1$, is positive. To determine c_i's BC scare, add $\sum_{j \in \mathcal{L}_1} x_{i,j}$ to the first summation. But $x_{ij} = -x_{ji}$, so

$$\sum_{i \in \mathcal{L}_1} \sum_{j \in \mathcal{L}_1} x_{i,j} = 0.$$

Thus, if $\sum_{j \in \mathcal{L}_1} x_{i,j} < 0$, the sum for another $\mathcal{L}_1$ candidate must be positive. Namely, at least one top-layer candidate has a positive BC score. In the unlikely case that each $\mathcal{L}_1$ candidate's sum from elections among other $\mathcal{L}_1$ candidates is zero, then all $\mathcal{L}_1$ candidates have a positive BC score. For candidates in the bottom layer, interchange "positive" and "negative." As some candidate from the lower layer has a BC negative score, the conclusion follows. □

Results of this type are relatively easy to discover because the summations used to determine the BC scores cancel cyclic effects of the Condorcet profile differentials. But consider the equally important concern of comparing *the relative standing of the BC winner and loser among the pairwise rankings.* This new issue is more difficult to analyze because it involves introducing – rather than removing – cyclic terms to the Basic profile. The increased complexity of the analysis probably explains why I have found no mention of this issue in the literature. Indeed, it is not even clear what to conjecture. However, the structure of the Condorcet terms as developed here allows several new conclusions; e.g., the following is the converse of Theorem 11.

Theorem 12. *If the pairwise ranking define a strict transitive ranking, the BC winner is ranked above the BC loser. More generally, if* $\mathbf{q}_n$ *defines a layer structure, the BC winner cannot be in a lower layer than the BC loser.*

The proof of Theorem 12 involves a symmetry condition (Lemma 1) showing that the BC winner and loser satisfy this privileged status with other ranking procedures. For instance, Theorem 14 was discovered (and its proof just uses a particular distance) with Lemma 1. Examples from Table 7.6 prove that attempts for a stronger version of Theorem 12 fail; e.g., we might suspect that the BC winner always is in layer strictly above that of the BC loser, but Region 14 provides counter-examples.

To see the intuition behind Theorem 12 and that even stronger assertions (which involve the tallies) exist consider the profile $\mathbf{p}^* = \mathbf{p}_B + t\mathbf{C}_r^n$ where Basic portion $\mathbf{p}_B$ defines the ranking $c_1 \succ c_2 \succ \ldots \succ c_n$. While the Condorcet term creates a cyclic affect, the actual $\mathbf{q}_n$ tallies are modified by the Basic tallies. As the largest Basic difference in tallies comes from the Basic (or BC) winner and loser, this larger value is reflected in the final tally; it is the most resistant to change. So, as long as the pairwise rankings are transitive, the BC winner is ranked above the BC loser.

More generally, if $x_{1,n} > 0$ for $\mathbf{p}^*$, then the assertion follows. If $x_{1,n} = x_{1,n}^B + x_{1,n}^C < 0$, where the superscript indicates the Basic and Condorcet values, then $x_{1,n}^C < -x_{1,n}^B < 0$. If the $x_{1,n}^C$ value comes from a $\mathbf{C}_r^n$ primary, then, because $x_{1,n}^B \geq |x_{j,k}^B|$ for any other pair, the rankings of all pairs from the primary cycle are completely determined by $\mathbf{C}_r^n$. Hence all candidates, including c_1 and c_n are in the top layer. If $x_{1,n}^C$ comes from a sth cycle of $\mathbf{C}_r^n$, then the difference for $\mathbf{C}_r^n$ tallies in earlier fundamental cycles is larger than $|x_{1,n}^C|$. Thus the same $x_{1,n}^B \geq |x_{j,k}^B|$ inequality demonstrates that the $\mathbf{C}_r^n$ tallies for pairs from the primary, second, …, sth cycles determine the pairwise rankings. In particular, all candidates are in the top-cycle. (The complete proof replaces the fundamental cycles with $\mathbf{R}_r^n$ rankings.) The remaining case is if $x_{1,n} = 0$ and this is due to the primary cycle. The same argument shows that the ranking "almost" includes the primary cyclic with the only difference being that $c_1 \sim c_n$. Again, there is only one layer.

So, only minimal relationships exist between the Condorcet and BC winners (also see Saari [17, 18]). By combining this observation with the profile structure, we extract new properties about Black's method. One uses the fact that a slight change in the t value in a profile $\mathbf{p}_B + t\mathbf{p}_C$ can cause the Condorcet portion to destroy the existence of a Condorcet winner by creating a cycle. This implies a severe discontinuity for Black's method; a discontinuity which manifests a sudden shift in emphasis from the Condorcet portion of a profile, $\mathbf{p}_C$, to the Basic portion $\mathbf{p}_B$. The following describes one of many admissible scenarios.

Theorem 13. *There exist profiles where, by adding a small number of new voters, the Black outcome drastically changes so that the previous winner now is ranked next to the bottom.*

7.2.3 Copeland

Copeland outcomes also depend on the Condorcet portion of a profile because the Condorcet portion determines which $\mathcal{RC}(n)$ ranking region contains $\mathbf{q}_n$. Copeland's method replaces $\mathbf{q}_n$ with the orthogonal cube vertex of this region and projects the vertex to the transitivity plane. (This projection is due to the summation part of the definition.) If $\mathbf{q}_n$ is in a transitive ranking region, then this ranking is the Copeland ranking. Thus, Figures 8 and 9 provide profiles illustrating BC and Copeland differences. For a profile in region 1, for instance, the cycle cancels terms leading to the Copeland ranking $A \sim B \succ C \sim D$. On the other hand, the region 16 Copeland ranking reverses this outcome to have $C \sim D \succ A \sim B$. Notice, for these profiles, Copeland's method ranks the Borda winner *below* the Borda loser. Also notice that many of the Copeland peculiarities occur because the cycles form a Copeland cancellation forcing the Copeland rankings to be determined by the rankings of the remaining pairs. (This observation simplifies the proofs and extends assertions from (Saari and Merlin [25].)

So, the Condorcet portion of a profile forces differences between the Copeland and BC rankings. Copeland and Black's method agree when the rankings are transitive; differences emerge when the rankings lose a Condorcet winner. Here, Copeland's method cancels cycles so the outcome is based on the remaining information. Black, on the other hand, reverts to the Basic profile outcome.

7.2.4 Kemeny

To understand Kemeny's method, consider profile $\mathbf{C}_r^n$. By symmetry, this profile is equally close to any of the n transitive rankings in $\mathbf{R}_r^n$ so the Kemeny outcome includes all n of these rankings. The Basic portion breaks this tie to determine the Kemeny outcome. Geometrically, the Basic portion of $t\mathbf{p}_B + \mathbf{C}_r^n$ moves $t\mathbf{q}_n^T + \mathbf{q}_n^C$ from the central $\mathbf{q}_n^C$ location in a $\mathbf{q}_n^T$ direction; this change must reflect the $\mathbf{q}_n^T$ properties. In particular, we should expect that $\mathbf{q}_n^T$ breaks the Kemeny ties in favor of the largest Basic profile difference – between the BC winner and loser. This is the case. (See the proof of Theorem 12.)

Theorem 14. *(Saari and Merlin [26]) A Kemeny winner always is BC ranked above a Kemeny loser. Conversely, the Kemeny method always ranks a BC winner above the BC loser.*

7.3 Arrow-Raynaud procedure

To further illustrate the strong role of the Condorcet differential, I compare the BC with a clever procedure (AR) introduced by Arrow and Raynaud [2] for multicritera decision making. It uses the *outranking matrix* $A = [a_{ij}]$ where a_{ij} is the vote c_i receives in a $\{c_i, c_j\}$ contest. The AR ranking is obtained with their *primal algorithm* which identifies the maximum in each row. (The maximum in the ith row identifies c_i's largest pairwise election outcome; the column identifies her competitor.) The candidate associated with the smallest value is designated as bottom-ranked. Delete the row and column of this candidate, and repeat the process with the reduced outranking matrix to identify the candidate second from the bottom.[5] Continue until all candidates are ranked.[6]

For instance, matrix

$$\begin{pmatrix} - & 40 & 62 & 48 \\ 36 & - & 76 & 62 \\ 14 & 0 & - & 40 \\ 28 & 14 & 36 & - \end{pmatrix} \tag{7.7}$$

defines the AR ranking $A \succ B \succ C \succ D$. As the sum of the row j entries is c_j's BC score, the BC ranking of $B \succ A \succ D \succ C$ conflicts with the AR ranking. This summation cancels the Condorcet portion of the profile, so all AR and BC differences are due to the AR's reliance on the Condorcet portion of a profile.

To analyze the AR procedure, let $a_{ij} = a(B)_{ij} + a(r_1)_{ij} + a(r_2)_{ij} + a(r_3)_{ij}$ be given, respectively, by the Basic, $\mathbf{C}^4_{r_1}$, $\mathbf{C}^4_{r_2}$, and $\mathbf{C}^4_{r_3}$ portions of a profile. As only these profile portions affect the pairwise rankings (Theorem 8), profile $\sum_{j=1}^{4} a_j \mathbf{B}^4_j + \sum_{j=1}^{3} \gamma_j \mathbf{C}^4_{r_j}$ defines matrix $A = A_B + \sum_{j=1}^{3} \gamma_j A_{r_j}$ where A_B and A_{r_j} are, respectively, the outreach matrices defined by the Basic profile and by $\mathbf{C}^4_{r_j}$. For instance, with coefficients $a_2 = 5, a_1 = 4, a_4 = 1, a_3 = 0$

$$\begin{aligned} A_B &= 6 \begin{pmatrix} - & a_1 - a_2 & a_1 - a_3 & a_1 - a_4 \\ a_2 - a_1 & - & a_2 - a_3 & a_2 - a_4 \\ a_3 - a_1 & a_3 - a_2 & - & a_3 - a_4 \\ a_4 - a_1 & a_4 - a_2 & a_4 - a_3 & - \end{pmatrix} \\ &= \begin{pmatrix} - & -6 & 24 & 18 \\ 6 & - & 30 & 24 \\ -24 & -30 & - & -6 \\ -18 & -24 & 6 & - \end{pmatrix} \end{aligned} \tag{7.8}$$

By being defined by the Basic portion, the A_B ranking must agree with that defined by the a_j values and, in turn (Cor. 2), with the BC and Basic positional rankings. Indeed, with these a_j choices the A_B ranking $B \succ A \succ D \succ C$ holds for all positional methods over all subsets of candidates. (For instance, the plurality and BC ranking of this profile for $\{B, C, D\}$ must be $B \succ D \succ C$.) To generate conflict, observe that the direction of the $\mathbf{C}^4_r$ pairwise cycle is determined by

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[5] In case of ties, candidates are selected randomly.

[6] This description makes it clear that the AR rankings are remarkably consistent as candidates are dropped.

r. So, to create a new profile with tally favoring A over B and C over D, let $r_1 = (A \succ B) \succ (C \succ D)$. Adding $\gamma_1 \mathbf{C}^4_{r_1}$ to the Basic profile changes the outreach matrix to

$$\begin{pmatrix} - & -6+2\gamma_1 & 24 & 18-2\gamma_1 \\ 6-2\gamma_1 & - & 30+2\gamma_1 & 24 \\ -24 & -30-2\gamma_1 & - & -6+2\gamma_1 \\ -18+2\gamma_1 & -24 & 6-2\gamma_1 & - \end{pmatrix} \tag{7.9}$$

where $\gamma_1 = 4$ generates the new and conflicting outcome $A \succ B \succ C \succ D$. By adding 38 to each entry to eliminate the negative signs (i.e., add an appropriate $\mathbf{K}^4$ multiple), the resulting 76 voter profile changes Matrix 7.9 into Matrix 7.7. In other words, the Arrow-Raynaud ranking of Matrix 7.7 differs from the BC ranking (and that of any positional method) because the AR method reflects properties of the Condorcet portion while (Theorem 8) the BC and other four-candidate positional methods do not. The following summarizes the general situation; the proof is immediate from the profile decomposition.

Theorem 15. *On Basic profiles, the BC and AR rankings always agree. There are profiles where the two rankings disagree, and all such examples are caused and completely explained by how the AR procedure treats the Condorcet portion of a profile.*

Thus, the Condorcet portion, and only this portion, forces differences between the AR and BC rankings. More generally, any and all differences between rankings of AR and any other procedure (based on pairwise tallies) depend on how they treat the Condorcet portion. Thus all illustrating examples must exhibit a strong Condorcet component.[7] Thus, it is not surprising to discover the strong Condorcet component in all illustrating profiles Arrow and Raynaud use to contrast AR with competing procedures.

8 Arrow and the loss of individual rationality

According to Section 7, any critique of AR, Kemeny's method, Black's approach, the Copeland method, the Condorcet winner, and any other method depending on pairwise tallies requires interpreting the pairwise vote of Condorcet profile differentials. The mystery created by this differential is that (by construction with the ranking disk) the voters' rankings of profile $\mathbf{R}^n_r$ offer no candidate an advantage over another; each candidate is in first, second, . . ., last place exactly once. This symmetry suggests that the natural societal ranking of the n candidates is a complete tie. Indeed, this tied outcome, which occurs for any n-candidate positional procedure (Theorem 8, Part 3), is so natural that any contrary ranking Λ must be justified. But this complete tie does not occur with pairwise voting,

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[7] To determine the Condorcet component in a $\mathbf{C}^4_r$ direction, add the number of voters with preferences from the Condorcet four-cycle defined by r and the number with preferences from the Condorcet cycle defined by $\rho(r)$. The difference between these sums reflects the strength of the $\mathbf{C}^4_r$ component.

so we must explain the outcome of pairwise cycles. As my argument extends that used in Saari [22] to discuss Arrow's Theorem, I start with Arrow's seminal conclusion.

8.1 Arrow's result

Transitivity is a sequencing condition which requires the pairwise rankings to mimic the ordering properties of points on the line. For instance, if a voter prefers $X \succ Y$ and $Y \succ Z$, then the voter must prefer $X \succ Z$. A voter with transitive preferences is called *rational*; a voter with non-transitive preferences is called *irrational.*

Arrow's Theorem uses the assumptions:

1. *(Unrestricted domain; rational voters)* Each voter has a complete, strict transitive ranking of the n alternatives; there is no restriction on the choice of the ranking.
2. *(Pareto).* If the voters are unanimous on the ranking of a particular pair of candidates, then that is the societal ranking of the pair.
3. *(Binary independence)* The societal ranking of each pair depends only on how each voter ranks the specified pair; the ranking of any other candidate is irrelevant for this ranking.

Arrow characterized all procedures satisfying these minimal conditions.

Theorem 16. *(Arrow [1]) Let there be $n \geq 3$ alternatives and $v \geq 2$ voters whose preferences satisfy the unrestricted domain condition. Let $\mathcal{F}$ be the set of ranking procedures with transitive outcomes which satisfy the Pareto and binary independence conditions. Each $\mathcal{F}$ procedure is equivalent to an identity mapping of a single variable; that is, it is a dictatorship where the societal outcome always agrees with the dictator's preferences.*

As this paper is not concerned with abstract procedures, I also consider other properties satisfied by the procedures studied here such as *anonymity*, where the outcome depends only on how many voters have each (strict) ranking of the candidates, and *neutrality,* where the candidates are ranked according to the number of assigned points. Not all of these procedures satisfy the next condition which extends Arrow's "binary independence" condition to k-candidate subsets.

Definition 10. *Let F_S be a procedure which ranks the candidates in a specified subset S. This procedure is* "S-consistent" *if its outcome depends only on how the voters rank the candidates in S. Namely, if $\mathbf{p}_1$ and $\mathbf{p}_2$ are two profiles where each voter has the same relative ranking of the candidates in S, then $F_S(\mathbf{p}_1) = F_S(\mathbf{p}_2)$. A procedure defined over all n candidates is said to satisfy the kth-level independence condition if it is S-consistent for each subset S of k candidates.*

With the four candidates $\{A, B, C, D\}$, a plurality election for the subset $S = \{A, B, C\}$ is S-consistent because the outcome depends only on how each voter ranks these three candidates; it is irrelevant how a voter ranks D. The procedure which finds the plurality ranking for each subset of three candidates satisfies the ternary independence condition.

8.2 Losing rationality with heterogeneity

My interpretation of the Condorcet differentials depends upon the following "partial transitivity condition."

Definition 11. *Let* $2 < k \leq n$. *A set of rankings of the* n *candidates is* k-level transitive *if there is a transitive ranking for each subset of* k *candidates.*

A transitive ranking of the n candidates is a k-level transitive ranking. Although the preferences

$$\{A \succ B \succ C,\ B \succ C \succ D,\ C \succ D \succ A,\ D \succ A \succ B\} \tag{8.1}$$

are not transitive, they are transitive at a three-level. What prevents a transitive ranking is that the voter's rankings for $\{A, C\}$ and for $\{B, D\}$ change depending on who is the "other" candidate. (For instance, the voter has $A \succ C$ when B is available, but $C \succ A$ when D is the other candidate.) In other words, rather than fixed binary rankings, this "conditional binary ranking" condition allows a voter's pairwise rankings to change with the subset of candidates. Such behavior is not unusual in actual human behavior (e.g., see the work of Tversky and Kahneman [29]), but it is prohibited by the definition of rationality.

The following obvious assertion is central for our analysis.

Theorem 17. *A procedure which is defined over transitive rankings and which satisfies a* k*th level independence condition also is defined over the set of all* k*-level strict transitive rankings.*

To illustrate, the voter characterized by Eq. 8.1 cannot vote in a BC election for four candidates, but he *can* vote in a BC election for any triplet. Similarly, because the irrational cyclic preferences $A \succ B$, $B \succ C$, $C \succ A$ satisfy the binary level of transitivity, this voter can vote in any pairwise election. Indeed, the three irrational voters defined by

$$\begin{cases} \text{Two prefer} & A \succ B,\ B \succ C,\ C \succ A, \\ \text{One prefers} & B \succ A,\ C \succ B,\ A \succ C \end{cases} \tag{8.2}$$

can use pairwise votes to obtain the expected and natural (for these preferences) cyclic election outcome $A \succ B$, $B \succ C$, $C \succ A$ where each election is decided by a 2:1 vote. Here, two voters have one belief while one has a directly opposite belief.

Theorem 17 shows that the domain for a procedure satisfying k-level independence *extends beyond rational voters to include irrational voters only capable of rationally ranking subsets of* k *alternatives.* Thus, "individual rationality" is a profile restriction. The issue we need to resolve is whether this restriction has any effect with procedures based on positional methods for k-candidates. Namely, when a procedure which satisfies anonymity and k-level independence is used with a sufficiently heterogeneous society, could the actual profile of rational voters be indistinguishable (to the procedure) from profiles which satisfy only a k-level transitivity condition? If so, then it is arguable that certain voting oddities reflect a procedure's attempt to capture the views of non-existent irrational voters rather than the actual rational voters. In other words, procedures which satisfy a k-level transitivity condition could vitiate the assumption of individual rationality of voters.

To see this is the case, reassign the binary rankings of the Eq. 8.2 preferences to voters to construct the three-voter Condorcet profile of rational preferences

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B. \tag{8.3}$$

Anonymity and binary independence prevent the pairwise vote from distinguishing the Eq. 8.3 profile from that of Eq. 8.2 where the cyclic outcome is a natural, expected conclusion. Indeed, the pairwise vote cannot distinguish the Eq. 8.3 profile from any other profile where the three rankings of each pair are reassigned, in any manner, among the three voters.

A simple computation proves that the binary parts define five distinct profiles. In addition to Eqs. 8.3, 8.2, the last three have two voters with opposing transitive rankings (so their pairwise votes cancel) while the third has the cyclic preferences $A \succ B, B \succ C, C \succ A$ which breaks the tie in favor of a cyclic outcome. Consequently, *any procedure satisfying anonymity and binary independence cannot distinguish between the rational beliefs of Eq. 8.3 and four other profiles involving irrational beliefs. As the natural outcome for 80% of these profiles is a cycle, the cycle is the natural outcome of the Condorcet triplet.* A cycle occurs for Eq. 8.3 because binary independence forces the procedure to ignore the crucial individual rationality assumption.

8.3 Sufficiently heterogeneous

Suppose a procedure (such as one satisfying a k-level of transitivity) cannot distinguish between rational and certain kinds of irrational preferences. Once society becomes sufficiently heterogeneous, parts of each rational profile can be reassembled to define an irrational profile with a "natural" non-transitive outcome. Namely, using certain procedures with a sufficiently heterogeneous society has the effect of weakening the rational agent assumption. It remains to characterize "sufficiently heterogeneous".

As shown next, a "sufficiently heterogeneous society" for the pairwise vote is characterized by the Condorcet profile differentials. A quick way to make this point is to prove that by excluding these differentials, Arrow's conditions allow non-dictatorial outcomes. In fact, the constraints can be made even more restrictive by adding conditions to Arrow's list, and natural, non-dictatorial methods still exist. Eliminating Condorcet terms is a profile restriction; all other profile restrictions I know about which avoid a dictator for Arrow's assertions work because they impose restrictions on the Condorcet portion of a profile. (One exception is the infinite agent setting. But the fact outcomes for the infinite agent setting radically differs from any finite agent case indicates that these theorems involve poor modeling assumptions, or are amusing exercises with limited meaning for choice theory (Saari [24]).)

Theorem 18. *For $n \geq 3$ candidates, let $\mathcal{NC}(n)$ be the subspace of profiles with no Condorcet component. There exist non-dictatorial procedures which satisfy the unrestricted domain condition on $\mathcal{NC}(n)$, Pareto, and binary independence. In fact, there exist procedures, such as the BC, Black's method, Kemeny's method, the Copeland method, and the pairwise rankings, which satisfy these conditions as well as neutrality and anonymity.*

Proof. On $\mathcal{NC}(n)$, only the Basic (Borda) terms affect pairwise rankings. The conclusion follows from properties of these differentials. The BC is the only positional method which satisfies these conditions because all other positional procedures are influenced by other profile differentials. □

The next statement asserts that parts from a Condorcet profile can be reassembled into a large number of other profiles which satisfy the k-level transitivity condition. Perhaps more important than the assertion is the proof for $k = 2$ as it describes how to characterize the representation cube.

Theorem 19. *Let k, $2 \leq k < n$, be an integer. The $\binom{n}{k}$ relative rankings from each transitive ranking of $\mathbf{R}_r^n$ can be reassembled to create a large number of profiles where each voter's preferences satisfy k-level transitivity, but only profile $\mathbf{R}_r^n$ consists solely of rational voters. For each subset of candidates, at least k of the voters' preferences define a k-candidate Condorcet cycle.*

Proof. Repeated applications of Theorem 8, part6, show that going from n to $n - 1$ candidates leaves a transitive ranking and a Condorcet cycle. Going from $n - 1$ to $n - 2$ leaves a Condorcet cycle and two added rankings where one ranking comes from the $(n - 1)$-Condorcet cycle and the other is the restriction of the previous extra ranking. Both rankings depend upon which candidates are dropped. The same argument holds after any number of candidates are dropped. The remainder of the proof is in Section 10. □

To illustrate with $\mathbf{R}_r^n$, $r = c_1 \succ c_2 \succ \ldots \succ c_n$, we now know that $\mathbf{R}_r^n$ is the only rational profile defined by the binary parts. Another class of profiles is where two voters have directly opposing views (so their votes cancel in pairwise votes) while the remaining $(n - 2)$ voters have cyclic preferences agreeing with the primary cycle defined by r. The number of such profiles is determined by the number of pairs of opposing preferences (that is, the number of opposing vertices of the orthogonal cube – remember, these preferences need not be rational) so there are at least $2^{\binom{n}{2}-1}$ of these profiles. (For $n \geq 5$, more irrational profiles are generated by the combinatoric number of ways to assign rankings from the sth level cycles to voters.) Thus, the election cycles is the natural outcome for more than $1 - [1/2^{\binom{n}{2}-1}]$ of the possible profiles that are indistinguishable with these pairwise characteristics. To illustrate with $n = 6$, far more than $16,385$ different profiles can be constructed from a Condorcet cycle; all but one involve irrational voters where, it is arguable, the cycle is an accurate outcome. The sole anomaly is the Condorcet profile where, it is arguable, that a complete tie is a more accurate conclusion. Since the cycle is the "correct conclusion" for over 99.994% of the profiles with these characteristics, it must be treated as the natural outcome.

In other words, the main and only difficulty with the pairwise vote is that it is incapable of strictly ministering to our intentions of servicing only the needs of a very rare minority of rational profiles. Namely, *combining the Condorcet cycle with the pairwise vote drops the individual rationality assumption.* To illustrate, the pairwise vote cannot distinguish between $\mathbf{R}^4_r$, $r = A \succ B \succ C \succ D$, and the following irrational profile.

Voters	$\{A,B\}$	$\{B,C\}$	$\{C,D\}$	$\{A,D\}$	$\{A,C\}$	$\{B,D\}$
$1-2$	$A \succ B$	$B \succ C$	$C \succ D$	$D \succ A$	$A \succ C$	$B \succ D$
3	$A \succ B$	$B \succ C$	$C \succ D$	$D \succ A$	$C \succ A$	$D \succ B$
4	$B \succ A$	$C \succ B$	$D \succ C$	$A \succ D$	$C \succ A$	$D \succ B$

(8.4)

The first and last voters have completely opposite rankings, so they define a tie that is broken by the cyclic preferences of voters 2 and 3. For $k = 3$, the following irrational profile comes from rearranging the triplets from $\mathbf{R}^4_r$.

1	$A \succ B \succ C$	$B \succ C \succ D$	$C \succ D \succ A$	$D \succ A \succ B$
2	$A \succ B \succ C$	$B \succ C \succ D$	$C \succ D \succ A$	$D \succ A \succ B$
3	$B \succ C \succ A$	$C \succ D \succ B$	$D \succ A \succ C$	$A \succ B \succ D$
4	$C \succ A \succ B$	$D \succ B \succ C$	$A \succ C \succ D$	$B \succ D \succ A$

(8.5)

While each imaginary voter in Eq. 8.5 has a transitive ranking for each triplet, the triplets are not compatible with any four-candidate transitive ranking. Indeed, irrational voter one's rankings of triplets define a cycle. (These are the "extra" rankings promised by Theorem 8 when a candidate is dropped from a Condorcet profile.) But three-candidate positional procedures cannot distinguish between the rational and partially rational voters, so a level of individual rationality is dismissed by using any positional method with triplets. In particular, for each subset, the preferences of voters 2-4 define a Condorcet triplet causing a positional tie vote which is broken by voter one. This defines a natural cyclic outcome over the rankings of the four triplets. Again, the Condorcet tuple weakens the assumption of individual rationality.

8.4 Consequences

This analysis provides answers for several mysteries about voting. For instance, as only Condorcet and Basic terms affect BC outcomes, we now know the source of all possible BC paradoxes caused by dropping candidates. (The same argument extends to all positional methods, but they are subject to other profile differentials which further distort the outcomes.)

Corollary 6. *The n-candidate BC ranking is not affected by the Condorcet portion of a profile; it agrees with the pairwise ranking of the Basic component of the profile. Any changes in the BC ranking when candidates are dropped are due to the Condorcet portion of the profile. Consequently, these changes – and the BC rankings of subsets – are influenced by a partial loss of transitivity in the profile.*

Let k satisfy $2 \leq k < n$. For each candidate, the sum of her BC tallies from a n-candidate Condorcet profile differential over all k candidate subsets is zero.

This result provides a strong argument to ignore the BC rankings of subsets and to place value on the BC ranking of all n-candidates. The reason is that by using the BC rankings of subsets, we dismiss the crucial information that the voters are rational. For an immediate implication, recall that Nanson method [12] is a run-off where, at each stage, the BC bottom ranked candidate is dismissed and the remaining candidates are reranked with the BC. The corollary suggests that the reranking process dismisses valuable information about the voters. Instead, we should keep and use the original BC ranking of all n-candidates.

These comments help us re-examine a standard argument which Brams [4] nicely captures with a simple example involving candidates A, B, C and X. His seven voter profile has three voters with preferences $C \succ B \succ A \succ X$, two with $B \succ A \succ X \succ C$ and two with $A \succ X \succ C \succ B$ defining the BC ranking $A \succ B \succ C \succ X$. Brams notes that by dropping bottom-ranked X, the BC ranking reverses to become $C \succ B \succ A$. In arguing that A, not C, should be top-ranked, he reflects a widely held belief that the BC behavior allowing A to vault to top place "when 'irrelevant' candidate X is introduced [illustrates] the extreme sensitivity of the Borda count to apparently irrelevant alternatives." However, according to Cor. 6, a changed BC ranking requires a strong Condorcet element which alters the triplet's ranking at the cost of weakening the assumption that voters are individually rational. Indeed, Brams' profile nearly completes $\mathbf{R}^4_r$ with $r = C \succ B \succ A \succ X$ (only $X \succ C \succ B \succ A$ is missing) where the Basic and Condorcet portions of this profile are

$$\frac{1}{24}\{[7\mathbf{B}^4_A + 6\mathbf{B}^4_B + 5\mathbf{B}^4_C] - 21\mathbf{C}^4_{A\succ B\succ C\succ X}\}.$$

The bracketed Basic term defines the natural ranking of $A \succ B \succ C \succ X$ – the original BC ranking – while the dominant Condorcet term undermines the assumption of individual rationality of the voters. Thus, the original BC ranking should be trusted.

The last assertion of Cor. 6 means that by summing the BC scores a candidate receives over *all* k-candidate subsets, we reobtain the BC outcome. (The summation cancels the Condorcet portion of the profile leaving only the influence of the Basic profile.) This assertion makes it straightforward to determine the admissible relationships among a BC ranking for the subsets of k-candidates as they reflect the effects of the Condorcet portion of the profile. (Because of the cyclic effects of the extra term obtained by dropping a candidate (see Theorem 8, part 6), the relationships involve a cyclic effect.) This provides simpler arguments than those used in Saari [18].

An extension answers a mystery concerning differences between the pairwise and BC rankings. Namely, by ignoring how rational voters sequence pairs in a transitive manner, the pairwise vote dismisses all information corroborating the individual rationality of voters.

Corollary 7. *Any and all differences between the pairwise and the BC ranking are caused by the Condorcet portion of a profile. Therefore, the pairwise ranking partially reflects a weakening of the assumption of the individual rationality of voters. Because any disagreement between the BC and Condorcet winners (losers or rankings) is due to the fact that Condorcet's approach is influenced by the Condorcet portion of a profile, the Condorcet outcome is influenced by the partial loss of the assumption of the individual rationality of the voters. Conversely, profiles can be constructed to illustrate any difference between the BC and Condorcet rankings by use of Basic and Condorcet profile differentials.*

To appreciate the consequences of Cor. 7, recall that the Condorcet winner is the standard for many in the field of social choice. Indeed, the *Condorcet principal* requires a procedure to elect the Condorcet winner when one exists. However, as Cor. 7 proves that the Condorcet winner is a flawed concept, we must wonder about the value of a procedure satisfying the Condorcet principal. Even stronger, *by satisfying the Condorcet principal, a procedure should be viewed with suspect* because its outcomes are influenced by phantom irrational voters. Similarly, because Black's method emphasizes the Condorcet winner, it places strong emphasis on the Condorcet portion of a profile. In turn, the outcome is influenced by the views of non-existent irrational voters.

This observation about the role of the Condorcet portion explains all flaws of all procedures using pairwise rankings. If the procedure does not cancel the Condorcet profile differential, then the outcome exhibits a bias — the portion of the outcomes from the Condorcet portion reflects a loss of individually transitive preferences. Using this observation, it now is easy to construct examples illustrating all possible cycles, or where the outcomes of a procedure fail to reflect the voters' true views. As in the description of the Arrow-Raynaud procedure, the analysis reduces to simple algebra. These comments are captured in the following formal statement.

Theorem 20. *For $n \geq 3$ candidates, if the ranking of Black's method, Kemeny's rule, the Copeland method, or the Arrow-Raynaud approach differ from the BC* *ranking, then the difference is completely due to the procedure's treatment of the Condorcet portion of the profile. Consequently, the difference is because the procedure allows a partial loss of the assumption of individual rationality of the voters.*

This assertion significantly simplifies the analysis in choice theory. Namely, if any procedure based on pairwise votes agrees with the pairwise rankings on the Basic profiles, then any and all differences with the BC are caused because the procedure includes information from the Condorcet portion of the profile. Consequently, all analysis can emphasize these aspects.

9 More candidates, more procedures

With $n \geq 5$ candidates, the Condorcet profile differentials span a space with dimension larger than $\binom{n-1}{2}$. The extra dimensions allow these differentials to influence not only pairwise and BC outcomes but also the plurality and other positional outcomes. Thus, we want to keep only the Condorcet portions which determine pairwise rankings but not non-BC positional rankings.

Definition 12. *For candidates c_i and c_j, the $c_i \succ c_j$ Condorcet profile differential is the sum of all Condorcet profile differentials determined by rankings r where the top and second ranked candidates are, respectively, c_i and c_j.*

To illustrate, the $A \succ B$ Condorcet profile differential for four candidates combines the two Condorcet profile differentials defined by $A \succ B \succ C \succ D$ and $A \succ B \succ D \succ C$. The following theorem asserts the $c_i \succ c_j$ Condorcet profile differentials does not influence the remaining election outcomes. The tallies associated with the the new differentials can be easier to use because they emphasize two candidates rather than several.

Theorem 21. *Assume there are $n \geq 4$ candidates.*

1. *Each $c_i \succ c_j$ Condorcet profile differential is orthogonal to the Basic (and Borda) differentials. The set of all Basic vectors and all $c_i \succ c_j$ Condorcet vectors determine all pairwise and all BC outcomes.*
2. *A $c_i \succ c_j$ Condorcet profile differential is orthogonal to all double reversal profile differentials. There exist symmetry changing profile differentials from $\mathcal{UK}^n$ which are not orthogonal to the $c_i \succ c_j$ Condorcet profile differential.*
3. *For the $c_i \succ c_j$ Condorcet profile differential, c_i beats c_j in a pairwise election with the $(n-2)(n-2)! : -(n-2)(n-2)!$ tally. However, c_j beats and c_i loses to all other candidates with a $(n-2)! : -(n-2)!$ tally. The pairwise outcome for any other pair of candidates is a tie where each candidate receives zero votes.*
4. *For a k-candidate subset, $2 < k \leq n$, the tally of a $c_i \succ c_j$ Condorcet profile differential is the same for all positional methods. If both c_i and c_j are in the set, then c_i receives $(n-k)(n-2)!$ points, c_j receives the negative of this,*

and all other candidates receive zero points. If c_i is in the set, but c_j is not, then c_i receives $-(k-1)(n-2)!$ points while each other candidate receives $(n-2)!$ points. If c_j is in the set when c_i is not, then c_j receives $(k-1)(n-2)!$ points and each other candidate receives $-(n-2)!$ points. For all other sets, all candidates receive zero points.

Think of the $A \succ B$ profile differentials as separating the Condorcet effects into a sum of Condorcet triplets. Namely, the $A \succ B$ Condorcet differential is the sum of Condorcet triplets $A \succ B, B \succ X, X \succ A$ where X ranges over the $(n-2)$ other candidates. The tally for a triplet has equal differences; because $\{A, B\}$ is in $(n-2)$ of the triplets, the sum enhances its tally.

The $c_i \succ c_j$ Condorcet profile differential offers at least two advantages. The first is that the profile does not create profile deviations for positional voting outcomes (part 4). A second advantage (part 2) is that the relative tallies for only a limited number of pairs of candidates are affected. So, to construct a profile where the BC ranking is $c_2 \succ c_3 \succ \ldots \succ c_{10} \succ c_1$ even though the BC rankings of $\{c_1, c_2\}, \ldots, \{c_1, c_2, \ldots, c_9\}$ reflect the $c_1 \succ c_2 \succ \ldots \succ c_9$ ranking, start with a Basic profile with the indicated ten-candidate BC outcome. Next, add appropriate multiples of $c_1 \succ c_j$ Condorcet profile differentials, $j = 2, 3, \ldots, 10$ to force the desired outcome for the different subsets.

10 Proofs

Proof of Theorem 2. The linearity of the tallying procedure ensures there is a kernel. The fact the universal kernel, determined by the pairwise and plurality votes, is in the kernel of all procedures is a direct consequence of Proposition 1. From the linearity of tallying, the $\mathcal{UK}^n$ dimension is the difference between the dimensions of the normalized space of profiles $(n!-1)$ and the normalized space of vote tallies.

The normalized space of vote tallies is where election tallies are replaced with the fraction of the total vote received by each candidate. As a k candidate election has $(k-1)$ degrees of freedom, the dimension of all pairwise elections is $\binom{n}{2}(2-1)$, of all triplets is $\binom{n}{3}(3-1), \ldots$. The total dimension is $\sum_{j=2}^{n}(j-1)\binom{n}{j} = \sum_{j=2}^{n} j\binom{n}{j} - \sum_{j=2}^{n}\binom{n}{j}$. It follows from Theorem 1 that this is the dimension of the image space for the plurality vote. To compute the summation, differentiate the equality

$$(1+x)^n = \sum_{j=0}^{n}\binom{n}{j}x^j \text{ to obtain } n(1+x)^{n-1} = \sum_{j=1}^{n} j\binom{n}{j}x^{j-1}.$$

By setting $x = 1$ and using algebra, the stated expression follows. □

Proof of Theorem 4. In a plurality or pairwise tally of $\mathbf{B}^n_{c_i}$ of a subset containing c_i, she receives one point for each ranking where she is top-ranked. There are $(n-1)!$

of them, so this is her tally. All remaining candidates are treated symmetrically, so each receives the same point total over the subset. As $\mathbf{B}^n_{c_i}$ is a profile differential, the sum of the candidates' tallies is zero. So, in a k-candidate subset with c_i, each of the other candidates receives $-\frac{(n-1)!}{k-1}$ votes. In a set where c_i is not a candidate, each candidate receives zero votes.

By use of the summation process defining the $\mathbf{b}^k$ outcome, it follows that c_1 in a k candidate subset receives $\frac{1}{k-1}(k-1)((n-1)!)$ votes. (The fraction normalizes the BC outcome, the $(k-1)$ term is the number of pairwise elections.) If c_1 is not in a set, then the BC tally for all candidates is zero. Thus the BC and plurality outcomes for all subsets is as stated. According to Proposition 1, the normalized tally for all procedures agree.

To prove the summation assertion, notice that in the $(n-1)!$ terms where c_i is top-ranked, c_j is bottom-ranked in precisely $\frac{(n-1)!}{n-1} = (n-2)!$ of them, $j \neq i$. Similarly, in the $(n-1)!$ terms where c_i is bottom ranked, c_j is top-ranked in precisely $(n-2)!$ of them, $j \neq i$. The conclusion now follows with a simple computation. □

Rest of the proof for Theorem 6. It remains to show that the points in the transitivity place satisfy additive transitivity and that no unanimity profile is in this plane. To prove the additive transitivity assertion, notice that if $j < k$, then the $x_{j,k}$ component is 1 in $\mathbf{T}^n_j$, -1 in $\mathbf{T}^n_k$, and 0 in other $\mathbf{T}^n_s$ vectors. With this choice of coordinate axis for the orthogonal and representation cubes, the $x_{j,k}$ component for $\mathbf{q} = \sum_{i=1}^n \alpha_i \mathbf{T}^n_i$ is $(\alpha_j - \alpha_k)$. The additive transitivity statement now follows from addition.

A unanimity profile is not in the transitivity plane if the pairwise tallies do not satisfy the additive transitivity condition. This is a trivial computation. □

Proof of Theorem 8. Part 1. This is a simple computation.

Part 2. By symmetry considerations, it suffices to prove this assertion for a specified pair, say $\{A, B\}$. Consider the profile differential $\mathbf{p}_{A\succ B}$ which has one voter for each ranking where $A \succ B$ and -1 voters for each ranking where $B \succ A$. The dot product of $\mathbf{p}$ with $\mathbf{p}_{A\succ B}$ gives $\mathbf{p}$'s $\{A, B\}$ tally. Therefore, if a non-zero multiple of $\mathbf{p}_{A\succ B}$ is in the space spanned by the Borda and Condorcet differentials, the assertion follows.

To prove that $\mathbf{p}_{A\succ B}$ is in this space, start with $\mathbf{p}_1 = \mathbf{Bor}^n_A - \mathbf{Bor}^n_B$. Consider a ranking where A and B are, respectively, k_1th and k_2th ranked. As the total number of voters with this ranking are $(n+1-2k_1) - (n+1-2k_2) = 2(k_2 - k_1)$, the number of voters depends upon the difference, Δ, in A-B rankings. Notice that each ranking with $A \succ B$ is paired with a ranking where $B \succ A$ generated by interchanging the A and B candidates. The difference is that if $2(k_2 - k_1)$ voters have the first ranking, then there are $-2(k_2 - k_1)$ voters with the second ranking.

I now show that $n\mathbf{p}_{A\succ B}$ can be constructed by adding appropriate Condorcet profile differentials to $\mathbf{p}_1$. In doing so, remember that either $\mathbf{C}^n_{r_1}$ agrees (up to sign) with $\mathbf{C}^n_{r_2}$, or the two differentials have no rankings in common. Also recall that every ranking is in some Condorcet profile differential. Because of the symmetry and because we are using profile differentials, in the following computations it suffices to keep track of the number of voters with each ranking where $A \succ B$.

Start with a ranking r_k where A and B are, respectively, top and $(k+1)$th ranked. This means that in each $\mathbf{C}^n_{r_1}$ ranking where $A \succ B$, A is ranked either k or $(n-k)$ candidates above B; the latter rankings come from the $\rho(r_k)$ portion. Thus, adding $\alpha_k \mathbf{C}^n_{r_k}$ to $\mathbf{p}_1$ adds α_k voters to the $2k$ voters for each of the rankings where A is ranked k candidates above B and subtracts α_k voters from the $2(n-k)$ voters where A is ranked $(n-k)$ places above B, $k \leq n/2$. By choosing $\alpha_k = n - 2k$, there are precisely n voters with each of these rankings. To complete the construction of $\mathbf{p}_{A \succ B}$, do the same for each ranking where $A \succ B$ that has not been adjusted.

Part 3. The total number of points in a positional method $\mathbf{w}^n = (1, w_2, \ldots, w_{n-1}, 0)$ is $\sum_{j=1}^n w_j$. Because each candidate is ranked first, second, ..., last precisely once in $\mathbf{R}^n_r$, each candidate receives $\sum_{j=1}^n w_j$ points. The only change in this argument for a candidate in the $\rho(r)$ portion is that each candidate receives $-\sum_{j=1}^n w_j$ points. This completes the proof.

Part 4. Let $\mathbf{C}^n_r$ be the Condorcet profile differential defined by r. To show that $\mathbf{C}^n_r$ is orthogonal to an arbitrarily chosen Basic profile $\mathbf{B}^n_{c_j}$, notice that the only terms they have in common is when c_j is top and bottom ranked in $\mathbf{R}^n_r$ and in $\mathbf{R}^n_{\rho(r)}$. In $\mathbf{R}^n_r$, these two rankings have the same number of voters; in the Basic profile, one term has a positive number of voters and the other has a negative number of these voters. Thus, these terms cancel. The same argument holds for the $\rho(r)$ portion.

To prove the statement about the double-reversal profiles, let the kernel profile be where there is one voter for each of the $r_1 \succ r_2$ and $\rho(r_1) \succ \rho(r_2)$ preferences and -1 voters for each of the $r_1 \succ \rho(r_2)$ and $\rho(r_1) \succ r_2$ rankings. Either $\mathbf{C}^n_r$ has no preferences in common (so they are orthogonal) with this double-reversal profile, or at least one preference shared by both profiles. Assume without loss of generality that one common preference is $r = r_1 \succ r_2$. Then, so is $\rho(r) = \rho(r_1 \succ r_2) = \rho(r_2) \succ \rho(r_1)$. As the number of voters with these two preferences agree in the double reversal profile, but differ by sign in $\mathbf{C}^n_r$, the scalar product of these terms cancel. It is easy to show that if $r = r_1 \succ r_2$ is a $\mathbf{C}^n_r$ ranking, then $r_1 \succ \rho(r_2)$ and $\rho(r_1) \succ r_2$ are not in $\mathbf{C}^n_r$. This completes the proof.

Part 5. The proof uses the fact that the Condorcet profile differentials partition the set of preferences. This partitioning occurs because, by construction, if r_1 and r_2 appear a Condorcet profile differential, then (up to sign of the number of voters with each preference), the profile differentials are the same. This proves that the sets (the orbits) are disjoint. That they fill the space follows from the fact that each ranking defines a Condorcet profile differential. For the dimension statement, notice that each Condorcet profile differential has $2n$ of the $n!$ preferences, so there are $\frac{1}{2}(n-1)!$ sets of these profile differentials with no preferences in common.

The proof for the setting where k is an even integer $2 < k < n$ only involves creating an example. This is done following the theorem.

Part 6. This is a direct computation. □

Cor. 3. The Basic profiles do not admit changes in rankings as candidates are dropped. Thus, all changes must come from the Condorcet portion. As shown, this occurs. □

Cor. 4. This is a direct computation. □

Theorem 10. Parts 1 and 2. This follows directly from Theorem 8 and the properties of the Basic profiles.

Part 3. This is immediate from the geometry of the representation cube.

Parts 4–6. These statements follow from the earlier description of the methods and their dependency on the pairwise tallies. □

Proof of Theorem 12. While each $\mathbf{C}^n_r$ has the fundamental cycle structure illustrated in Figure 5, sums of these differentials need not. (In a $c_1 \succ c_2$ Condorcet differential, certain cycles defined by one $\mathbf{C}^n_r$ are cancelled by another $\mathbf{C}^n_{r_1}$.) Yet, symmetry allows the fundamental cycles of pairwise rankings to be replaced with similar cycles of $\mathbf{R}^n_r$ transitive rankings; i.e., the primary cycle of $\mathbf{C}^n_r$ is replaced by the n $\mathbf{R}^n_r$ rankings. (For the $c_1 \succ c_2$ Condorcet differential, the primary cycle is replaced with all transitive rankings from all $\mathbf{R}^n_r$ where r has $c_1 \succ c_2$ as the two top ranked candidates.) The sth level cycles define appropriate rankings which, in turn, define the needed $\mathbf{R}^n_r$ replacements, etc. The proof of Theorem 12 and its many extensions exploit the fact that this symmetry structure holds for the wide variety of distances described next.

Definition 13. *Let $\delta(\mathbf{q}^1_n, \mathbf{q}^2_n)$ be a distance defined over the differences in pairwise tallies or pairwise rankings. Assume that δ satisfies a neutrality condition where the outcome depends only on differences in rankings and/or tallies, but not the identity of the candidates in each pair. The distance from $\mathbf{q}^1_n$ to a set of points $\mathcal{S}$ is*

$$\inf_{\mathbf{q}\in\mathcal{S}} \delta(\mathbf{q}^1_n, \mathbf{q}).$$

Let $\mathcal{C}$ be a collection of rankings. For each $r \in \mathcal{C}$, let $r(\mathbf{q}_n) = \{\mathbf{q}|$ the ranking assigned to $\mathbf{q}$ is r and $\mathbf{q}$ is obtained from $\mathbf{q}_n$ by reversing the signs of appropropriate $x_{i,j}$ values.$\}$ Then

$$\delta(\mathbf{q}_n, \mathcal{C}) = \inf_{r\in\mathcal{C}} \left[\inf_{\mathbf{q}\in r(\mathbf{q}_n)} \delta(\mathbf{q}_n, \mathbf{q}) \right] .$$

As examples, if δ is the usual Euclidean distance and $\mathcal{S}$ is the transitivity plane, then the $\mathcal{S}$ tally defining $\delta(\mathbf{q}_n, \mathcal{S})$ is the Basic tally. If $\mathcal{C}$ is the set of strict, transitive rankings where δ is the l_1 distance, then a nearest ranking is the Kemeny ranking. If $\mathcal{C}$ is the collection of rankings with a Condorcet winner and δ is the l_1 distance, then the minimal ranking is the Dodgson ranking.

For an example using pairwise rankings, let $sign(x_{i,j})$ be 1 if $x_{i,j} > 0$, -1 if $x_{i,j} < 0$, and zero otherwise. Let $\delta^{sign}(\mathbf{q}^1_n, \mathbf{q}^2_n) = \sum_{j<k} |sign(x^1_{j,k}) - sign(x^2_{j,k})|$.

If $\mathcal{S}$ is the transitivity plane, the ranking minimizing $\delta^{sign}(\mathbf{q_n}, \mathcal{S})$ is the Copeland ranking.

The following lemma asserts that any such δ has symmetry properties similar to those of the $\mathbf{C}^n_r$ fundamental cycle tallies of Figure 5.

Lemma 1. *Let $\delta(-,-)$ be a distance from Def. 13. Let r and r^* be any two strict, transitive rankings of the $n \geq 3$ candidates. Let $r_1, r_2 \in \mathbf{R}^n_{r^*}$ (i.e., they are any two of the n rankings in the Condorcet n-tuple defined by r^*) where the difference in tallies between the kth and jth ranked candidates from either ranking agree; $k < j$. Let $\mathbf{q}(\mathbf{C}^n_r)$ denote the pairwise tallies defined by $\mathbf{C}^n_r$. Then,*

$$\delta(\mathbf{q}(\mathbf{C}^n_r), r_1) = \delta(\mathbf{q}(\mathbf{C}^n_r), r_1). \tag{10.1}$$

Proof. Because the assertion is for all choices of metrics, I must show that the differences between $\mathbf{C}^n_r$ and r_j tallies and rankings of pairs is precisely the same for each j. Without loss of generality, let $r_1 = c_1 \succ c_2 \succ \ldots \succ c_n$ and $r_2 = c_j \succ c_{j+1} \succ \ldots \succ c_1 \succ \ldots \succ c_{j-1}$; i.e., the difference between r_1 and r_2 is j rotations in the ranking disk.

A uniform way to measure the differences in tallies and rankings is to determine how many and what kinds of binary rankings from $\mathbf{C}^n_r$ need to be reversed to make the ranking compatible with r_1 and r_2. By carrying out this computation with fundamental cycles, the argument is straightforward. This is because each fundamental cycle which includes the kth ranked candidate from r_1 can be identified (by a rotation of the ranking disk) with a fundamental cycle of precisely the same type that includes the kth ranked candidate from r_2.

Select two identical pairs of fundamental cycles (i.e., both are primary or both are sth level) where each includes the kth ranked candidate from the appropriate ranking. The next step is to compare the differences in rankings and tallies between pairs from the first cycle and r_1 with a similar analysis for the second cycle and r_2. By symmetry, the differences and agreements are identical (but with different names of candidates for the pairs). As this is true for all fundamental cycles, and as this includes all possible pairs, the same data about differences in rankings and tallies of pairs occurs between $\mathbf{C}^n_r$ and r_1 as between $\mathbf{C}^n_r$ and r_2. This is the only data admitted by the distances, so the conclusion follows. ☐

Denote the Condorcet portion of the profile

$$\mathbf{p}_{BC} + \sum_j \mu_j \mathbf{C}^n_{r_j}, \quad \mu_j \neq 0, \tag{10.2}$$

by $\mathbf{p}_C$ and its tally by $\mathbf{q}_C$. According to the lemma, for any r^*, each $r \in \mathbf{R}^n_{r*}$, and $\mu_j \mathbf{C}^n_{r_j}$, it takes the same number of changes of tallies and rankings from each fundamental cycle of $\mu_j \mathbf{C}^n_{r_j}$ to make the rankings compatible with r. Since this is true for each $\mu_k \mathbf{C}^n_{r_k}$, $k = 1, \ldots,$ it follows from the component-wise linear structure of vector addition that for each r^*, the number of changes of rankings and/or tallies needed to convert the rankings of $\mathbf{q}_C$ into r^* is precisely the same as for any r in $\mathbf{R}^n_{r*}$. Thus, if for a choice of δ we have that r^* is the kth closest transitive ranking (for any choice of k), then the same status applies to all $r \in \mathbf{R}^n_{r*}$.

The Basic portion breaks this remarkable $\mathbf{q}_C$ symmetry by moving the tally, $\mathbf{q}_{BC} + \mathbf{q}_C$, in the $\mathbf{q}_{BC}$ direction. But by breaking symmetry, not all $\mathbf{R}^n_{r*}$ rankings are equal distance from the new tally; the rankings that now are closer reflect the $\mathbf{q}_{BC}$ components; e.g., its largest component defines the difference between the Basic (BC) winner and loser. To see how this geometry proves part of the theorem, consider the plane passing through $\mathbf{q}_C$ which has $\mathbf{q}_{BC}$ as a normal vector. It follows from the geometry that if $r_1, r_2 \in \mathbf{R}^n_{r*}$ are on opposite sides of this plane where r_1 is on the $\mathbf{q}_{BC}$ side, then r_1 is the closer of the two in Euclidean (and many other norms[8]) to $\mathbf{q}_{BC} + \mathbf{q}_C$. Since the largest change in a component is due to the differences in the Basic (BC) winner and loser, if r_1 and r_2 are transitive, then r_1 has the BC winner ranked above the BC loser. In particular, if the pairwise tallies define a transitive ranking, it has the BC winner ranked above the BC loser.

Now suppose that $\mathbf{q}_{BC} + \mathbf{q}_C$, $t = 1$, defines a layer structure with at least two layers. If $x_{1,n} > 0$, the assertion follows. More generally, assume there are at least two layers. Using the l_1 norm (actually, any l_p, $1 \leq p < \infty$), we wish to find the transitive ranking closest to this layer structure. Clearly, this closest ranking must straighten out the rankings of each layer independent of what happens among layers. But, according to the definition of the layer structure (where that tallies are such that each candidate is a higher layer is ranked above all candidates in a lower layer) it follows immediately from the triangle inequality that in a nearest transitive ranking, all candidates from a higher layer remain ranked above all candidates from a lower layer. So, if the Basic or BC winner is ranked in a layer strictly below the BC loser, the nearest transitive rankings have the BC loser ranked above the BC winner. This contradiction proves the assertion. □

Rest of the proof of Theorem 19.. It remains to prove that when the $\binom{n}{k}$ relative k-candidate rankings from $\mathbf{R}^n_r$ are reassembled to create profiles where each voter's preferences satisfy k-level transitivity, the only transitive profile is $\mathbf{R}^n_r$. To introduce the basic ideas, start with $k = 2$ and the orthogonal cube. Each of the n rankings from $\mathbf{R}^n_r$ defines a vertex. Because of the relationship between the choice of the vertices, the convex sum of these vertices is the normalized pairwise tally of $\mathbf{R}^n_r$. Thus, this point, denoted by $P_{2,r}$, can be described in terms of the fundamental cycles; it is the $\frac{n-2}{n}$ multiple of the convex sum of the $(n-2)$ vertices where all have the binary rankings of the primary cycle, $(n-1)-2$ of them have the binary rankings of the $s = 2$ secondary cycle and 1 has the opposite rankings, ..., $(n-1)-s$ of them have the binary rankings of the sth fundamental cycle while s of them have the opposite binary rankings, $s \leq n/2$. Let $Q(n)$ be the number of different ways vertices can be selected to have these properties.

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[8] This includeas norms that require equality for the triangle inequality along a straight line. The straight line here connects $\mathbf{q}_{BC}$ with $\mathbf{q}_C$.

This construction means that to find other choices of preferences for n voters which give identical pairwise outcomes as $\mathbf{R}^n_r$, just add any two other diametrically opposed vertices (which cause a cancellation) to the above construction. There are $2^{\binom{n}{2}-1}$ such pairs, so there are at least $2^{\binom{n}{2}-1}Q(n)$ profiles involving irrational voters with the same pairwise outcomes.

Denote the vector from the origin to $P_{2,r}$ by $\mathbf{P}_{2,r}$. By construction, all n transitive vertices from $\mathbf{R}^n_r$ are on the $[\binom{n}{2}-1]$-dimensional plane passing through $P_{2,r}$ with normal vector $\mathbf{P}_{2,r}$. The "top" and "bottom" sides of this plane are, respectively, the side defined by $\mathbf{P}_{2,r}$ and by $-\mathbf{P}_{2,r}$.

Lemma 2. *The n vertices corresponding to $\mathbf{R}^n_r$ rankings are on the plane passing through $P_{2,r}$ with normal vector $\mathbf{P}_{2,r}$. The remaining $n!-n$ transitive vertices are on the bottom side of this plane.*

It follows from Lemma 2 that the convex sum of any n transitivity vertices, where at least one does not represent a $\mathbf{R}^n_r$ ranking, is on the bottom side of the plane. Thus, the pairwise tallies generated by $\mathbf{R}^n_r$ cannot be achieved with any other profile with transitive preferences.

Proof. Proof of Lemma 2. Assume (without loss of generality) that $r_1 = c_1 \succ c_2 \succ \ldots \succ c_n$ is a a $\mathbf{R}^n_r$ ranking. The importance of the following argument is that it indicates how to determine the structure of the orthogonal cube relative to this plane. Namely, with slight modifications, we can find all vertices above the plane. In doing so, this provides a description of the representation cube. Namely, for each r, the plane defined by $\mathbf{R}^n_r$ is one of the faces of the representation cube. Associated with each $r_1 \in \mathbf{R}^n_r$ is an adjacent transitive ranking obtained in the manner described below; this ranking defines another $\mathbf{R}^n_{r*}$ and another face of the representation cube.

For each orthogonal cube vertex, there are $\binom{n}{2}$ adjacent vertices obtained by reversing one binary ranking. Indeed, two adjacent vertices define an edge of the orthogonal cube which is parallel to the direction defined by the pairwise change. If the vertex is a transitive vertex, then there are precisely $(n-1)$ adjacent transitive vertices; they must be obtained by transposing any *adjacent* pair of candidates in the ranking. (For instance, the $(n-1)$ transitive rankings adjacent to r are where one of the $c_1 \succ c_2$ ranking, *or* $c_2 \succ c_3$ ranking, *or* …, $c_{n-1} \succ c_n$ is reversed. Trivially, a change of a pair that is not adjacent creates a non-transitive ranking.) So, from each transitive vertex, $(n-1)$ of the adjacent vertices are transitive and $\binom{n}{2} - (n-1) = \binom{n-1}{2}$ of them are not.

If a coordinate axis direction has a zero dot product with $\mathbf{P}_{2,r}$, then that $\mathbf{P}_{2,r}$ component is zero. According to the pairwise sums of the fundamental cycles of $\mathbf{R}^n_r$, this orthogonality occurs iff n is even and the direction corresponds to a binary from one of the $s = n/2$ degenerate fundamental cycles. Thus, the only vertices on the plane adjacent to the $r_1 \in \mathbf{R}^n_r$ vertex involve reversing the c_j and $c_{j+n/2}$ rankings in r_1; as this change involves a non-adjacent r_1 pair, the resulting adjacent ranking is not transitive.

The same argument proves there are vertices adjacent to r_1 both above and below the plane. For an adjacent vertex to be above the plane, a binary from r_1 must be reversed so that it now agrees with the ranking of that binary in the appropriate $\mathbf{R}_r^n$ fundamental cycle. But all adjacent r_1 binary rankings are represented and agree with the associated binary ranking in the primary cycle. Thus any change of a adjacent r_1 binary reverses how this binary is ranked in $\mathbf{P}_{2,n}$. Thus, all transitive vertices adjacent to r_1 are below the plane.

Next I generalize this argument to prove that for each transitive vertex, either it is on the plane and represents a $\mathbf{R}_r^n$ ranking, or it is below the plane. For ranking r, let $\mathbf{v}(r)$ denote the r-vertex. To convert r_1 into another specified transitive ranking, called the *target ranking*, we need to compute the needed binary changes. Changing a $\{c_i, c_j\}$ ranking just changes the $x_{i,j}$ sign; the magnitude of the change is 2. Thus all binary changes needed to convert r_1 into the target ranking $r*$ define a composite vector $\mathbf{v}(r*) - \mathbf{v}(r_1)$, with ± 2 entries. Whether $\mathbf{v}(r*)$ is above, on, or below the plane is determined, respectively, by whether the sign of the dot product

$$(\mathbf{v}(r*) - \mathbf{v}(r_1), \mathbf{P}_{2,n})$$

is positive, zero, or negative.

How a binary ranking changes the value of the dot product comes from the pairwise tallies of the $\mathbf{R}_r^n$ fundamental cycles. To illustrate with a special case, consider the binary changes needed to convert $r_1 = c_1 \succ c_2 \succ \ldots \succ c_n$ into the next $\mathbf{R}_r^n$ ranking of $r_2 = c_2 \succ \ldots \succ c_n \succ c_1$ by successively moving c_1 down through the rankings. The $c_2 \succ c_1$ and $c_n \succ c_1$ changes involve binary rankings from the primary cycle, but in opposing directions relative to $\mathbf{P}_{2,n}$. (The first is opposite the $\mathbf{P}_{2,r}$ $x_{1,2}$ direction; the second binary change agrees with the sign of the $x_{1,n}$ direction of $\mathbf{P}_{2,n}$.) The difference in sign and the fact that the magnitude of these components in $\mathbf{P}_{2,n}$ agree means that these values cancel. Similarly, the $c_s \succ c_1$ and $c_{n+1-s} \succ c_1$ change come from the sth level cycle but in opposite directions, so, again, there is a cancellation in the dot product of this vector change with $\mathbf{P}_{2,n}$. Since the composite vector can be expressed in terms of these symmetric terms, the composite vector has a zero scalar product with $\mathbf{P}_{2,r}$, $\mathbf{v}(r_2)$ is on the plane.

This zero dot product required a symmetric cancellation where, in successively moving c_1 through the ranking, the negative values (meaning the binary changes differ from those in $\mathbf{P}_{2,n}$) are for the first $n/2$ candidates encountered in the ranking disk arrangement, while the positive values come in a symmetric manner from the last $n/2$ candidates. Consequently, if target ranking only involves moving c_1 partway through r_1, then the scalar product with $\mathbf{P}_{2,n}$ is negative; thus the target ranking vertex is below the plane.

This argument shows that a way to compute $(\mathbf{v}(r*), \mathbf{P}_{2,n})$ is to compute the number of binary ranking changes that need to be made on the fundamental cycles so that the resulting rankings define $r*$. Clearly, at least one ranking from each cycle needs to be reversed and the rankings need to be done so that some one candidate is top ranked, some candidate is second ranked, ..., some candidate is bottom ranked. To illustrate with $r_1 = c_1 \succ c_2 \succ c_3 \succ \ldots \succ c_n$ and $r* \not\in \mathbf{R}_r^n$, precisely one primary ranking (the $c_n \succ c_1$ ranking) needs to be reversed so that this cycle is compatible with r_1, but at least two primary rankings need to be reversed for $r*$. (If only one were reversed, it would define a $\mathbf{R}_r^n$ ranking.) For $n = 4$, this is all that is needed in the computations as the remaining cycles are degenerate. For $n \geq 5$, there is either one (if n is odd) or two $s = 2$ cycles. In either case, only the two secondary binary rankings $c_{n-1} \succ c_1$ and $c_n \succ c_2$ must be reversed. (To see this, start with c_1 and look at the rankings of every other candidate; this is compatible with the secondary cycle except in the indicated places.) For $r*$ and c_j, since one candidate is preferred to c_j and c_j is preferred to one other candidate in this cycle, at least two binary rankings must be changed to make $r*$ transitive. Thus, the number of changes from the $s = 2$ cycles for $r*$ is greater than or equal to those for r_1. This handles all transitive rankings for $n \leq 6$.

More generally, for $n \geq 7$ and a particular $s > 2$, there are at least s binary changes needed so that they are compatible with a transitive ranking. Precisely s changes are needed for the rankings to be compatible with r_1 (they are $c_n \succ c_s, c_{n-1} \succ c_{s-1}, \ldots, c_{n-s+1} \succ c_1$.) Therefore, the number of sth level binary changes needed to be reversed to be compatible with $r*$ is at least this number.

The number of binary changes at each level needed to make the rankings compatible with r_1 is less than or equal to the number needed for $r*$, so $(\mathbf{v}(r_1), \mathbf{P}_{2,n}) \geq (\mathbf{v}(r*), \mathbf{P}_{2,n})$. For equality, only one primary binary can be reversed for $r*$; i.e., $r* \in \mathbf{R}_r^n$. This completes the proof. □

A way to handle the $k > 2$ setting is to create a cube for k-level rankings. This cube has $n!/2((n-k)!)$ coordinate directions where each $[-1, 1]$ interval correspond to a particular strict ranking of the k candidates and the reversal of this ranking. Thus, this cube has $2^{n!/2((n-k)!)}$ vertices of which $n!$ are unanimity vertices. Replacing the "orthogonal cube" is the convex sum of vertices with a single ranking for each k-subset. (The coordinates for the other rankings are set equal to zero.) The transitive rankings from $\mathbf{R}_r^n$ define n of these vertices; their convex sum defines $P_{k,n}$, which the origin and this point define $\mathbf{P}_{k,n}$. By symmetry of $\mathbf{R}_r^n$, the components of $\mathbf{P}_{k,n}$ also enjoy a symmetry; these symmetries define the kth level fundamental cycles. The rest of the proof now is essentially the same as above.

As an alternative way to see this assertion, notice that the k level rankings are based on an appropriate sequencing of the binary rankings. Hence, if the assertion were false, then one could show it also is false for $k = 2$; this would be a contradiction. (This sequencing is used to connect the different k-level orthogonal cubes into a polytrope which can be used to extract other properties.) □

Theorem 21. Part 1. Each Condorcet profile differential is orthogonal to each Basic profile, so the $c_i \succ c_j$ profile differential (a sum of Condorcet profile differentials) also is orthogonal to the Basic profiles. Similarly (part 4), all n-

candidate positional tallies of such a differential are zero. The last assertion follows from Theorem 8.

Part 2. It is shown in Theorem 8 that a Condorcet profile differential is orthogonal to a double-reversal profile, so the same assertion holds for the $c_i \succ c_j$ profile differentials. It suffices to provide an example to prove that there are symmetry changing profile differentials that are not orthogonal to a $A \succ B$ Condorcet profile differential. One such example is where $r = (C \succ D \succ A) \succ (B \succ E)$ where the other three rankings come from $\sigma_1(C \succ D \succ A) = C \succ A \succ D$. In this setting, only one of the four rankings in the symmetry changing profile is in the $A \succ B$ profile differential, so orthogonality is impossible.

Part 3. This is a simple computation involving the tallies from Theorem 8.

Part 4. This is a direct consequence of parts 1, 2 and the computations from Theorem 8. Because these computations are not difficult for the normalized borda and the plurality vote, these computations provide an alternative proof for part 1 when k is even. □

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