

Unsettling aspects of voting theory[★]

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Summary. Voting procedures are known to be plagued with a variety of difficulties such as strategic voting, or where a voter is rewarded with a better election outcome by not voting, or where a winning candidate can lose by receiving more support. Once we know that these problems can occur, the next objective should be to completely understand how, why, and where they arise. Namely, for each election procedure, the new goal is to determine when such problems can occur, all voter types who can cause these difficulties, and the actions they must take. This paper develops an easily used approach to handle all of these issues for standard voting methods. New intuitive explanations for these various oddities follow from this approach.

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1 Introduction

When a winning candidate receives added voter support, the extra votes should not cause her to lose, but it could. If a voter fails to vote, this neglect should not assist one of his personally more favored candidates to win, but it can. If a candidate's performance in a debate inspires voters to rank her higher in their preference ranking, this should not hurt her election chances, but it may. All of this and much more can occur (for insightful surveys see Nurmi, 1999, 2002).

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Since strategic behavior is still another setting where a change in a profile can create a surprising change in the societal outcome, it is clear that these concerns are of practical interest. Which procedures suffer these problems? For instance, the controversy accompanying the 2000 US presidential election has focussed attention on the “instant runoff” procedure. Is this so-called reform method an improvement? As shown here, this method suffers all of these deficiencies.

Examples illustrating these different oddities are in the books of Nurmi (1999, 2002); Riker (1982); and Saari (1995), in the papers of Fishburn (1982); Fishburn and Brams (1984) (which made early contributions to this subject), and in many other references. For the most part, however, the literature concentrates on whether these behaviors can occur. A distinct weakness of this “existence” emphasis is that it does not address whether these disquieting peculiarities can be dismissed as being essentially coccocted, isolated anomalies, or whether they identify issues which constitute likely violations about how our election procedures should behave.

These are important, practical concerns. So it is important natural next objective to go beyond establishing the existence of these curious behaviors to characterize all situations where they arise. Namely, once it is known a procedure experiences specific problems, the next pragmatic issue is to describe when, how, and why all of this can happen. As a natural illustration, the important Gibbard (1973); Satterthwaite (1975) result proves that all reasonable procedures involving three or more alternatives admit profiles where someone can be strategically successful. The next step, then, should be to identify all strategic settings, all agents who can be strategic in each setting, and all strategic options for each procedure.

Clues about how some of this might be accomplished – at least for strategic voting – can be found in various places in the literature. For instance, Kalai and Muller (1977) characterize domains of nondictatorial social welfare functions and nonmanipulable voting methods. The papers of Gärdenfors (1976); Blair (1981) single out certain profiles which can cause strategic voting problems. But, very little is known about the general circumstances that are conducive to strategic voting, monotonicity violations, etc. Very little is known about how all of these issues are related. This general question is addressed here.

Stated in practical terms, we need an easily used method to determine whether a specified procedure does, or does not, experience these bothersome behaviors. If a procedure is vulnerable to certain difficulties, the approach must identify all situations where these counter-intuitive behaviors can arise, all individuals whose actions can cause these problems, and all actions they can take to cause these problems. The approach should be sufficiently intuitive to facilitate the creation of illustrating examples. Developing such a method and then applying it to standard election methods is the purpose of this paper. In this spirit, the new results offered here identify the “when, how, and why” of these peculiar properties for classes of election methods. Another goal is to use this more accessible methodology to replace the mystery about these behaviors with an intuition about when, where, and why they occur.

Saari (1994, 1995) created a mathematical approach which completes much of this program. With Merlin, he applied this methodology to analyze Copeland’s (1996); Kemeny’s (2000) procedures; later, with a related but more assessible ap-

proach, he addressed the several problems which plague Approval Voting (Saari, 2001). A weakness of his approach, however, is that often it uses mathematical tools more advanced than typically employed in this area. Rather than just his papers, this seems to be a standard problem; as Nurmi's survey (2002) and his references make clear, the technical nature of most papers in this area appears to hinder a full appreciation of the results. Thus, for many people, this literature fails to provide the needed insight, intuition, or an accessible methodology. In this manner, the message that can be extracted from Nurmi's books (and his comments in personal correspondence) inspired the development of the current approach.

2 Basic notion

The approach is surprisingly simple. To introduce it, consider the standard three voter profile ABC , BCA , CAB where ABC means that a voter has A , B , and C , respectively, top, middle, and bottom ranked. Let the selection procedure be an agenda where the majority winner of an $\{A, B\}$ election is advanced to a majority vote comparison with C to determine the overall winner. Here, A wins the first election by 2:1, but she loses the second one to C .

To prevent his bottom ranked C from winning, the ABC voter could strategically vote for his *second-ranked* B in the first election instead of his preferred A . This strategic vote advances B to the second stage where she beats C to become the overall winner. Thus our strategic voter forces the personally preferred outcome of B rather than C .

The sole purpose of this exercise is to point out that the change from the sincere to a strategic profile forces the first election result to cross the $A = B$ tie to select B rather than A . The important observation is that *for each phenomenon described in the introductory paragraph to occur, an outcome must "cross" a tied result*. After all, if A wins, but B wins with a new scenario, then an election outcome passed over a $A = B$ tie.

The fact that a "tied election" implicitly accompanies each of these election oddities leads to the following simple program which can be used to analyze all of the above topics, and several others, for any specified election method.

1. For a given procedure, describe all scenarios leading to the election of a specific candidate.
2. For each scenario, describe all "nearly tied vote" settings; these settings identify where the final conclusion may change depending on how the tie is broken.
3. For each nearly tied vote, describe the consequences accompanying all ways voters could vote to break the tie.

Finding all strategies for all agendas

What makes this approach particularly simple to use is that only potential election outcomes – not the profiles – are involved. Indeed, with the above agenda, notice that the scenarios which elect C – if $\{A$ beats $B\}$ and $\{C$ beats $A\}$, or if $\{B$ beats

$A\}$ and $\{C \text{ beats } B\}$ – make no mention of profiles. Profiles, and the creation of examples, are handled later by using results from Saari (1999, 2000, 2001).

To illustrate this approach, I completely analyze all strategic behavior associated with the above agenda. Since the first step requires selecting a particular scenario, start with the one where C is elected because $\{A \text{ beats } B\}$ and $\{C \text{ beats } A\}$. The second step is to identify all “nearly tied” settings for this scenario. One possibility is a nearly tied CA sincere vote. The final step is to determine all consequences of a changed outcome; that is, if A rather than C wins. But for A to win at this stage, voters sincerely preferring C to A would have to vote for A rather than C . As this vote is counterproductive, we can assume it will not happen. Consequently, the final stage offers no strategic opportunities.

The analysis reduces to considering the remaining possibility of a nearly tied AB sincere outcome for the first election. The third step requires examining what happens if the near AB tie is broken in different ways; i.e., consider what happens if B wins. If B wins and C beats B , then nothing new has occurred. So, consider where a strategic vote breaks the AB tie in favor of B , and B beats C to become the overall winner.

The strategic election of B is personally better only for voters preferring B to C . In order to strategically vote for B rather than A in the first election, the voters must sincerely prefer A to B . This observation completely classifies all strategic voters; they must prefer A to B and B to C , so they must have ABC preferences. The analysis shows that their strategy is to advance a stronger candidate to the last stage.

In the agenda setting, only voters with ABC preferences – voters whose least preferred candidate is both the sincere winner and the candidate seeded last – have a strategic opportunity to elect their second ranked candidate. The sole opportunity occurs with a nearly tied AB outcome at the first stage and if the sincere pairwise rankings define a AB, BC, CA cyclic outcome. With the strategic goal of advancing a stronger candidate to the final stage, the only strategy is to vote for B in the first stage.

This description describes *all possible profiles* which support this strategic action. To analyze the remaining “ $\{B \text{ beats } A\}$ and $\{C \text{ beats } B\}$ ” scenario, just interchange the A and B names in the above.

Other agenda winners

Instead of C winning, A can win with this agenda only if $\{A \text{ beats } B\}$ and $\{A \text{ beats } C\}$; i.e., A wins with the agenda only if A is a *Condorcet winner*¹. (The analysis where B wins with this agenda is the same.) As the final pairwise election in the agenda never offers strategic opportunities, our approach requires us to consider nearly tied AB outcome for the first election. Strategically breaking this tie in favor of B makes either B or C the overall winner. If B is the strategic winner, then only

¹ A candidate who beats all other candidates in majority vote pairwise elections is called a *Condorcet winner*.

voters sincerely preferring BA would benefit by being strategic. Here there is a problem; to strategically vote for B rather than A , a voter must sincerely prefer AB , but these voters do not. Thus, in this setting, no strategic vote is possible.

No strategic opportunities for the agenda exist if A is the sincere Condorcet winner and B would beat C – the last seeded candidate – in a majority vote.

For later reference, notice that in this case the sincere AB and proposed strategic BA rankings define the transitive ABC and BAC rankings.

It remains to consider what would happen if C , rather than B , is the strategic winner (so, CB is the sincere majority vote outcome). Here, voters preferring C to A have an incentive to be strategic. But strategically voting for B over A in the first election requires sincere AB preferences. Therefore, only voters who prefer CA and AB , or those with CAB preferences, have strategic opportunities. Again, the analysis provides the strategic rationale; by voting for their bottom ranked B in the first election, a candidate the voters know will lose to C , they are strategically advancing a weaker opponent to the final stage. To summarize:

With A as a Condorcet winner, only voters with CAB preferences – those who have the last seeded candidate top-ranked – have a strategic opportunity, and it is to elect their top choice. The opportunity requires their second ranked candidate, A , to be the sincere Condorcet winner with a nearly tied AB outcome. The only strategy is to vote for their last choice of B in the first stage. While the sincere profile has a Condorcet winner, the strategic profile generates a pairwise BA, AC, CB cycle.

Although elementary, this analysis completely characterizes for any three candidate agenda all possible strategic settings, it identifies all possible kinds of voters who can be strategic in each setting, and it describes all of their strategic options. Notice the accompaniment of strategic opportunities with cycles (for either the sincere or strategic profile); as I show a bit later, for election procedures based on pairwise rankings, cycles always accompany actions and the other voting peculiarities. An interesting corollary of this fact is that if the assertion made by many authors (e.g., Van Deemen, 1999; Tangian, 2000) that it is unlikely for cyclic outcomes to occur in practice is accurate, then their claim also means that strategic voting and the other election oddities should not be expected.

Other kinds of behavior

What adds value to this surprisingly simple analysis is that all of the phenomena indicated in the introductory paragraph can be completely analyzed in the same manner. This can be illustrated with the agenda scenario where C wins because A , not B , won the first election. Notice, if the AB outcome changes because voters did not vote, the neglectful voters must prefer AB . If the resulting outcome of B beating C is personally preferred by these voters, then they prefer BC . Thus, if voters preferring ABC forget to vote, these neglectful voters are rewarded with a personally preferred result.

Notice how this approach connects and relates all of the voting peculiarities. Namely, after identifying all tie vote situations for the different scenarios, just identify who wins or loses by breaking the tie in various ways – the different winners and losers determine the different election oddities. The main bookkeeping task is to keep track of the different options.²

3 Strategic action

To further illustrate this methodology, I now use it to identify all strategic opportunities, all strategic voters associated with each opportunity, and all strategic actions for a variety of standard election methods. As ties involving several pairs are highly unlikely, only settings with one nearly tied outcome are considered. (This approach, however, can also be used to analyze profiles with any number of ties; some conclusions differ.) Also assume that only small numbers of voters try to be strategic. While this standard assumption reflects traditional game theoretic discussions, it also avoids the need to justify how and why we should accept large scale, carefully coordinated strategic behavior.

Positional methods

Riker (1982) calls “positional voting” those procedures where specified weights are assigned to a candidate depending on her “position” (or ranking) on each ballot. The plurality vote assigns one point to a voter’s top ranked candidate, and no points for any other candidate. The Borda Count is where the difference between successive weights is the same positive constant. For five candidates, the standard option assigns 4, 3, 2, 1, 0 points, respectively, to a voter’s top, second, third, fourth, and last ranked candidate. Any choice of weights defines a procedure if it satisfies the obvious constraint that a lower ranked candidate does not receive more points than a higher ranked one. So, with $N \geq 2$ candidates, let w_j be the weight assigned to a j th ranked candidate where $w_j \geq w_{j+1}$; let $w_N = 0$.

While there are papers which analyze aspects of strategic voting with the plurality vote and Borda Count, to the best of my knowledge, only three-candidate strategic settings have been fully characterized for all positional methods (Saari, 1994, 1995). Yet, the above approach makes it surprisingly easy (almost trivial) to describe all possible strategic opportunities for all possible positional methods involving any number of candidates.

Without loss of generality, always let $C_1 C_2 \dots C_N$ be the sincere positional election ranking of the N candidates. According to our program, to determine whether an different candidate could be strategically elected, we must assume a nearly tied $C_1 C_2$ outcome. Of course, only voters preferring C_2 to C_1 would entertain strategic action. Because a successful strategy must move the outcome across a

² Thus, this approach can be used to understand whether one kind of oddity implies another. Incidentally, I have been told that aspects of this method are reminiscent of the familiar tool of backwards induction; a tool which has been widely used in social sciences’ e.g., Farquharson (1969); McKelvey and Niemi (1978) among others have used it to study strategic voting.

Table 1. Strategic options permitted by positional methods

		BAC	BCA	CBA
		$w_1 - w_2$	w_1	w_2
BAC	$w_1 - w_2$	None	$w_2 \neq 0$	$2w_2 > w_1$
BCA	w_1	None	None	None
CBA	w_2	$w_1 > 2w_2$	$w_1 \neq w_2$	None

tie, all strategies must involve marking the ballot to increase the C_2, C_1 differential to favor C_2 . This is summarized as follows:

If a voter’s sincere preferences are such that the positional method assigns the maximum number of points to C_2 and zero points to C_1 , then that voter has no strategic options. All other voters can be strategic. A strategic vote is one which increases the C_2 differential over C_1 .

Since a two candidate election has $w_1 = 1, w_2 = 0$ weights, this observation improves as a special case the known conclusion that a two candidate majority vote election never has strategic opportunities.

Three candidates

To analyze three candidate elections, let the sincere election outcome be ABC with a nearly tied AB tally. The objective is to determine whether this tally provides a strategic opportunity for some voter type to elect B . The types of voters who would benefit from strategic action (those who prefer B to A) are listed in the first column of Table 1; the differential their vote provides for B over A is listed in the second column. All potentially strategic ways to vote, which require increasing the differential provided to B over A , are listed in the first row with the associated $B-A$ differential listed in the second row.

The analysis now is simple. For each sincere voter type just check which strategic votes provide a larger $B-A$ differential. Should this never happen, a “None” is inserted in the matrix entry; e.g., the second row shows that a voter with BCA preferences (so B and A receive, respectively, the maximum and minimum weight values) has no strategic options. If a strategic vote is successful only with certain positional outcomes, the choice of procedures is listed in the appropriate matrix entry. For instance, since a voter with CBA preferences sincerely provides a w_2 differential for B over A , voting strategically as though BAC creates a $w_1 - w_2$ differential between B and A . This strategy is effective only if $w_1 - w_2 > w_2$, or if $w_1 > 2w_2$. This $w_1 > 2w_2$ restriction on strategic procedures is in the appropriate matrix entry.

To summarize, all strategic opportunities to influence the choice of the winner require a near tie between the two top ranked candidates. Table 1 specifies that the strategic voters must sincerely prefer the societal second ranked candidate over the sincere winner, and they must *not* have extreme views about the societal two

top ranked candidates; i.e., they cannot have one top ranked and the other bottom ranked. (The second row shows that the *BCA* voters, with the extreme views where *B* and *A* are top and bottom ranked, have no strategic options. The other two voter types have strategic options depending on the procedure.) Notice how all of the voting strategies replace the voter's moderate sincere views relating *A* and *B* by strategically voting in a more extreme manner.

Extracting more information

For other information coming from this table, notice that with the plurality vote the best a strategic voter can do is to avoid electing his bottom choice. For all other positional methods, however, voters who have the sincere winner second ranked can strategically elect their top choice. The reason for this mysterious difference among procedures is explained later.

A count in Table 1 of the usable strategies shows that the plurality vote and "vote for two" methods each have one strategic voter type with two strategies, while the Borda approach has two strategic voter types with one strategy each. All other positional methods have three different strategies spread between two strategic types. A word of caution; this count need not indicate which procedures are more susceptible to manipulation. For instance, a more delicate technical analysis (Saari, 1990, 1995) proves that the Borda Count is the least susceptible while the plurality vote is the most susceptible to strategic manipulation.

When a strategic voter can select among several strategies, a "better" strategy is one which provides a larger point differential. To illustrate, if a *BAC* voter uses the the *BCA* strategy, the differential between the strategic and sincere vote is $w_1 - (w_1 - w_2) = w_2$ points while the *CBA* strategy has a $w_2 - (w_1 - w_2) = 2w_2 - w_1$ differential. As w_2 always is larger than $2w_2 - w_1$ for $w_1 \neq w_2$, the more effective strategy is to vote as though *BCA*.

More candidates

A $N \geq 4$ candidate analysis requires a larger matrix, but the analysis remains straightforward. With the sincere positional outcome $C_1 C_2 \dots C_N$ outcome, the strategic actions associated with a near $C_1 = C_2$ tie follow:

- A voter who sincerely has C_2 top ranked has no strategic opportunities with the plurality vote; the best a strategic voter can do with the plurality vote is to avoid electing a personally lower ranked candidate. All other procedures offer opportunities to strategically elect a top-ranked candidate; e.g., if C_2 and C_1 are personally first and second ranked, then keep C_2 top-ranked and strategically vote the more extreme position where C_1 is bottom ranked. For procedures where $w_2 > w_3$, a strategic option is to vote as though C_1 is third ranked. If $w_2 - w_3 > w_1 - w_2$, this voter could vote as though C_2 is second ranked, and C_1 is third ranked, etc.
- All other settings are analyzed in the same manner.

In summary:

A positional election with a sincere $C_1 C_2 \dots C_N$ outcome, $N \geq 3$, always allows strategic opportunities for some voter type if there is a close $C_1 C_2$ vote. The voters with strategic opportunities are those who sincerely prefer C_2 over C_1 in a manner where the procedure does not assign the maximum number of points to C_2 and the minimum number to C_1 . (Thus the plurality vote never allows a voter to strategically elect a personally top-ranked candidate, but all other positional methods provide strategic opportunities to assist a most preferred candidate.) All strategic behavior requires voting to increase the point spread offered to C_2 over C_1 . So, voters with extreme views (relative to the election procedure) of the two top ranked candidates have no strategic options; for voters with more moderate views can strategically vote as though having more extreme beliefs (relative to the procedure).

Effects of extreme vs. moderate views

This last comment provides intuition about the source of the Gibbard-Satterthwaite Theorem for positional voting (actually, for all procedures). With only two candidates, each voter must have extreme views of the alternatives relative to the majority vote because one candidate is top ranked and the other is bottom ranked. Thus, nobody can be strategic. But adding candidates creates voter types with more temperate views; the voters with more temperate views are the only ones who can be strategic. The Gibbard-Satterthwaite result, then, is one consequence of the diversity of voter types which emerge with three or more candidates. This added variety is what ensures that situations must occur where some voter type has a strategic opportunity. As we will discover, this “extreme vs. moderate views” behavior accompanies all strategic action, but it need not always involve the societal top ranked candidates.

To further illustrate this “extreme vs. moderate” comment, notice that if a voter preferring C_2 to C_1 does not have C_2 top ranked, his view of C_2 and C_1 is “moderate” relative to the plurality vote. (This is because both candidates receive the same number of points from the voter – zero.) Consequently, this voter can to strategically adopt an “extreme position” by voting as though C_2 is top ranked. But, should C_2 be personally top ranked, this voter has “extreme” $C_2 - C_1$ views with the plurality vote, so the voter has no strategic options.

As all other positional methods assign a positive weight to a second ranked candidate, these procedures designate voters with C_2 top ranked and C_1 second ranked as having “moderate views.” Thus with these procedures these voters can strategically adopt a more extreme view. Again, these scenarios require at least three candidates to generate enough voter types.

This argument also shows that *any way to rank C_2 above C_1 is a strategic vote for some positional method and some sincere profile*. While it is natural to wonder whether strategic actions are limited, after subtracting off the voter’s sincere preferences and ignoring how other candidates are ranked, it follows that there can

be $\frac{N(N-1)}{2} - 1 = \frac{(N-2)(N+1)}{2}$ possible strategic ways to vote where the strategies vary over positional methods and sincere preferences. Rather than rare, strategic actions are abundant.

Multistage elimination procedures

By knowing how to determine all strategic actions for all positional elections, it is much easier to analyze methods which involve several positional outcomes. Consider, for instance, “multi-stage elimination procedures” where at each stage a certain number of top ranked candidates of a specified positional method are advanced to a next stage; the overall winner is the winner of the last stage.

A natural example is a *positional-election runoff* where the two top ranked candidates in a specified positional election are advanced to a majority vote runoff. With three candidates, a plurality runoff becomes the *Hare* method (where the candidate with the least number of first place votes is dropped at each stage) while the “vote-for-two”-runoff becomes *Coombs’ method* (where the candidate bottom ranked by most voters at each stage is dropped). *Nanson’s method* (Nanson, 1882) is a multi-stage procedure which advances only candidates who receive more than the “average” number of assigned Borda points at each stage. Other multi-stage methods are *tournaments*, where each stage involves specified pairings of alternatives and the majority winner is advanced to a next stage, and *agendas*, where each stage consists of a single pair and the winner is matched with a specified alternative at the next stage.

All sorts of other methods can be envisioned. A nine candidate procedure, for example, could partition the candidates into three sets of three candidates. Each set could be ranked with specified positional methods where the winner is advanced to the final set of three. In the final set of three, the winner of a specified positional method is the overall winner. A closely related method is the US presidential election where the different political parties hold primary elections and the winners are advanced to the general election. This generalizes to procedures where, at preliminary stages, the candidates are partitioned into subsets; each voter selects a particular subset and the winners are advanced to a next stage.

Our approach allows us to identify all strategic opportunities and actions for all of these procedures. For instance, if a multi-stage procedure advances three or more candidates to the last stage for a positional election, the appropriate strategies for this final stage are described above. Practical illustrations come from US presidential plurality elections where cries of “don’t waste your vote” are used to encourage supporters of minor candidates to strategically vote for a particular major candidate.

All remaining strategies are similarly analyzed; they influence who is advanced to a next stage. As indicated with the agenda, this approach shows that the strategic objective may be

1. to advance a weaker candidate who would be beaten at a later stage, or
2. to advance a stronger candidate who can win at a later stage.

Whichever situation, the strategy is to vote in a more extreme manner than an voter sincerely believes. This argument applies to all multi-stage methods; e.g.,

a traditional runoff, or if the voters must select a particular preliminary election. Thus this includes US presidential elections when voters vote in the “other party’s primary” to try to advance a weaker candidate for the general election.

Strategic opportunities and cyclic election outcomes

Before discussing specific procedures, a useful new observation is introduced. It asserts that certain successful strategies must be accompanied by cyclic rankings. Recall, the standing assumption is that only one pair is near a tie.

Suppose a successful strategic action exists for a multi-stage procedure which influences which of two candidates, say B or C , is advanced to a later stage. There is some candidate A so that the relative strategic CB ranking, or the relative sincere BC outcome, at this stage creates a cycle when compared with the sincere $\{A, B\}$, $\{A, C\}$ relative rankings at a later stage.

To illustrate this statement, recall that one case in the earlier agenda analysis required both the sincere AB and strategic BA rankings to define transitive rankings with C . The above statement supports what was discovered earlier; this particular A, B change cannot be strategic. As an immediate application of this statement, since a two-candidate election never admits strategic action, this “cyclic” behavior must accompany all strategies for any multi-stage method which has a pair at the final stage. This includes runoff elections and procedures such as agendas and tournaments which involve pairwise comparisons at each stage. With Nanson’s method, strategic settings require the sincere or strategic votes to have A Borda ranked above B for some set of candidates, in a later subset, B is Borda ranked above C , but had A been advanced, A would be Borda ranked above C . As another application, this statement extends the earlier comment that if such cyclic behavior really is unlikely to occur in practice, then a successful strategic manipulation of multistage methods is unlikely.

For the proof, first consider where candidate C is strategically advanced because she can beat a later candidate A , but the sincere winner B could not. The obvious strategy, then, is to reverse the relative sincere BC outcome in an earlier stage to CB . Thus, the sincere relative rankings from the different sets are BC, CA, AB , or cyclic. Consequently,

strategically advancing a stronger candidate to defeat a future candidate requires a sincere cyclic relative ranking of some triplet A, B, C , but a transitive strategic ranking.

The second strategic action is where at a later stage A would lose to the sincere winner B but not to the strategically advanced C . Since A will beat C but not B at a later stage, the strategic CB relative outcome generates the relative AC, CB, BA cyclic rankings. Consequently,

if the strategic goal is to advance a weaker candidate to help another candidate win at a later stage, the sincere relative rankings (coming from the

rankings of the relevant different sets of candidates) of A, B, C are transitive, but the strategic relative outcomes are cyclic.

This completes the proof.

Notice that this argument does not involve profiles nor even any specifics of strategic voting; it only describes how changes in outcomes influence later comparisons. Consequently, it follows that *with multi-stage procedures, this cyclic behavior must accompany all of the different phenomena described in this paper*, such as where a candidate loses by receiving more votes, or a voter is rewarded with a better outcome by not voting, etc.

Instant vs. standard runoff

Can a procedure be modified to make it less susceptible to strategic behavior? This suggests comparing the strategic features of the “standard positional runoff” which has two separate elections where the runoff involves the two top positionally ranked candidates, with the “instant runoff” where the ballots for the first election are used to tally the runoff election. The goal is to determine any strategic differences.

Using the same approach, assume that the sincere positional ranking is ABC . With the exception of the Borda Count, it turns out (Saari, 2001) that this ABC ranking imposes no constraint on the pairwise election rankings; the pairwise rankings could be anything, they could even reverse the positional outcome or be cyclic. What simplifies our analysis is the above statement which requires cyclic outcomes to accompany any successful strategic behavior. More precisely, according to the above, if A can be elected by strategically advancing a weaker candidate C to the runoff, then the relative CB strategic positional outcome must define a cycle with the sincere $\{A, B\}$ and $\{A, C\}$ pairwise rankings. The only way this this strategic behavior can occur, then, is if the sincere pairwise outcomes are BA and AC .

Following the methodology, suppose there is a nearly tied AB tally for the positional outcome. As breaking this tie in any manner still advances the same two candidates to the runoff, this setting provides no strategic opportunities.

Table 2. Strategic options to advance a weak candidate

		CAB	CBA	ACB	BCA
		w_1	$w_1 - w_2$	w_2	$-w_1 + w_2$
ABC	$-w_2$	All	All	$w_2 \neq 0$	$w_1 < 2w_2$
ACB	w_2	$w_1 \neq w_2$	$w_1 > 2w_2$	None	None
CAB	w_1	None	None	None	None

Next assume a nearly tied BC sincere tally in the first vote. Since the strategic goal is to elect A rather than B , the three types of voters who might consider strategic behavior are listed in the first column of Table 2 where the second column lists the sincere $C-B$ differentials. A strategic vote must increase the $C-B$ differential, so all

possible options are listed in the top row of this table with the respective strategic differentials listed in the second row. Again, the analysis just requires comparing the sincere and strategic differentials to determine which strategic votes increase the differential. If this happens with all procedures, an “All” is placed in the appropriate matrix entry; if this holds only for certain procedures, the restrictions on the w_j weights are specified; if it never happens, then “None” is listed. For instance, a sincere ACB voter voting as though CBA can help C only if $w_1 - w_2 > w_2$, or $w_1 > 2w_2$; this is the matrix entry.

Again, new information is provided by Table 2. First, when the strategic goal is to advance a weaker candidate to the runoff, all voters with strategic options can elect their top-ranked candidate. Indeed, only the two kinds of voters with the strategic winner top-ranked have strategic options. To illustrate this strategic behavior with the special case of a presidential primary, suppose A will represent the Orange party, while the Blue party has a primary to select between B and C . Voting in the Blue party’s primary carries the implicit assumption that the voter does not have A top-ranked; thus, Table 2 includes the strategy of an Orange party member voting in the Blue primary to elect a weaker opponent to face the Orange party’s choice of A . An interesting twist on this strategy occurred during the 2002 California gubernatorial primaries when the Democratic incumbent G. Davis mounted a strong ad campaign to ensure the defeat of his potentially strongest Republican opponent in the Republican primary.

It is informative to compare the elections and positional runoffs. In a positional election, only voters with moderate views about the sincere and strategic winner can be strategic. The conclusion changes for a runoff. For instance, Table 2 shows that a CAB voter with moderate A - B views about the eventual winner has no strategic options. To explain, while the intent of the strategy is to influence the eventual winner, the actual strategy targets who is advanced to the runoff. Thus only voters with moderate views (relative to the positional procedure) about who should be advanced to the runoff can strategically vote a more extreme position. This is precisely what Table 2 indicates.

Also notice that even after ignoring the BCA strategy, a runoff admits more strategies than a simple positional vote. In a positional vote, the plurality, Borda, and “vote for two” approaches each allow two strategic entries; here each allows four. All other positional runoffs have five strategic matrix entries.

To see the difference between a standard and instant runoff, notice that in a standard runoff, a strategic voter can vote for C over A in the first stage, but then switch to vote for A over C in the runoff; e.g., a voter can strategically vote CAB in the positional election and then vote sincerely for A over C in the runoff. (This is precisely what happens when a voter votes in the other party’s primary.) In an instant runoff, however, the CAB and CBA strategies cast a runoff vote for C against the voter’s best interest of A . If it is not clear whether the runoff will be decisive, such a strategic vote could be counterproductive; this is particularly so for the ABC voter who could help elect his bottom ranked C .

As Table 2 indicates, only voters with moderate views about who is advanced to the final stage *and* about the sincere and strategic winner can be strategic with an instant runoff when there is doubt about the runoff outcome. Since the interpretation

of moderate vs. extreme views depends on the choice of the positional voting procedure, this comment means that the *ABC* voter has no strategic options with an instant plurality runoff.

In a runoff election where the goal is to advance a weaker candidate to the runoff, a voter with extreme views between which of the two should be advanced has no strategic opportunities; other voters do for certain procedures. When the strategic goal is to advance a weaker candidate to the runoff and there is doubt about the margin of victory in the pairwise runoff, an instant runoff provides fewer strategic opportunities and options than a standard runoff.

It is interesting that the conclusions change when the strategic goal is to advance a stronger candidate to defeat *A*. An example with this flavor comes from the 2000 presidential primaries when some Democrats, who were uninterested in either of their major candidates, voted for Senator McCain in the Republican primary as he was their preferred candidate.

Here, our connection relating cyclic rankings and successful strategic behavior requires the relative strategic *CB* positional ranking to be transitive when combined with the sincere pairwise rankings, while the sincere *BC* ranking must create a cycle. Consequently, the sincere pairwise rankings must be *CA* and *AB*; so *C* is the strategic winner. The voter types contemplating this strategy (they prefer *C* to *A*) are listed in the first column of Table 3 with the sincere *C-B* differentials in the second column. All matrix entries identify which strategies are productive for which procedures. For instance, *CAB* voters, with extreme views about who is advanced to the runoff, never have a strategic option, while a *BCA* voter, with more moderate views both about who is elected and who is advanced to the runoff, has the most strategic options. The earlier count from Table 2 about strategic entries still applies.

Table 3. Strategic options to advance a strong candidate

		CAB	CBA	ACB	ABC
		w_1	$w_1 - w_2$	w_2	$-w_2$
CAB	w_1	None	None	None	None
CBA	$w_1 - w_2$	$w_2 \neq 0$	None	$2w_2 > w_1$	None
BCA	$w_2 - w_1$	All	$w_1 \neq w_2$	All	$2w_2 < w_1$

Contrary to what occurs with positional elections, Table 3 shows that a voter with extreme views between the strategic and sincere winner (the *CBA* voter) not only has strategic options in a runoff, but he can strategically elect his top choice. On the other hand, this voter has no strategic opportunities with a plurality runoff. So, while a strategic voter can elect a top ranked candidate by strategically advancing a weaker candidate, when a plurality runoff is being used the strategy of advancing a stronger candidate only avoids electing a personally bottom ranked candidate. Namely, when advancing a stronger candidate to a runoff, all strategic voters must

have the sincere winner bottom ranked; this differs from when the goal is to advance a weaker candidate.

While Table 3 identifies all strategies to advance a stronger candidate for a standard runoff, it turns out that with the instant runoff the ACB and ABC strategies are potentially counterproductive because they end up voting for A , rather than C , in the runoff. By using the instant runoff, then, these strategic votes are eliminated. But for all positional methods, the CAB strategy dominates the ACB or ABC strategies by providing a larger C - B differential and the CAB strategic vote is supportive of the strategic voter's interests in the instant runoff. Thus, when the goal is to advance a stronger candidate, there are no real strategic differences between an instant and regular positional runoff.

When the strategic goal is to advance a stronger candidate to the runoff, a voter with extreme views between the two options of who is to be advanced cannot be strategic. Other voters can do so with certain procedures. With the instant or regular plurality runoff, the strategic voter cannot elect his top-ranked choice. For all positional runoffs, there are no effective differences in the strategies available to a standard and an instant runoff system.

So, for runoff elections, all strategic action involves the second and third ranked candidates with the goal of influencing who is advanced.

General multistage elimination procedures

The same message about strategic action extends to all multistage elimination procedures. Consider, for instance, the strategic goal of advancing a weaker candidate C because at a later stage A can beat C but not the sincere B . If the strategic voter truly prefers AB , then to strategically vote for C over B , the voter either must prefer BC or have moderate B - C views relative to the procedure. If the earlier stage involves a pairwise comparison between B and C , then only ABC voters, with moderate B - C views can be strategic. (An example is an agenda where A is the last seeded candidate.) If the earlier strategic target involves several candidates, then, depending on the positional method being used, the voter could even have CB preferences and be strategic. All of this is reflected in Table 2. When extending Table 2 to any number of candidates, we now have that the relative sincere ranking of the strategic voters have the strategic winner ranked above the two candidates involved in the strategic action.

If the strategic goal is to advance a stronger candidate C to beat A where the strategic voter truly prefers CA , then, to strategically vote for C over B , the voter either prefers BC or has moderate B - C views relative to the positional method. The first case requires a relative BCA sincere preference; it is illustrated by an agenda where last seeded A would beat B but lose to C . The second case is illustrated by Table 3. In general, strategic voters of this kind have relative rankings where the sincere winner is bottom ranked in the relative ranking of this triplet.

Other subtle properties

Rather than being strategic just for personal gain, strategic behavior may be needed to allow a societal more acceptable election outcome. To illustrate with the 2000 Florida election to elect presidential electors, Bush narrowly won the plurality outcome even though it is believed that enough Nader voters had Gore second ranked so that Gore would have beaten Bush in a head-to-head election. This makes it arguable that a “Don’t waste your vote” strategy of voting for Gore rather than Nader might have produced an outcome more accurately reflecting the wishes of most Florida voters.

So, when can a strategic outcome for a procedure elect a societal preferred outcome? For measures of “good” and “bad” outcomes, it is traditional to use the Condorcet winner and loser³ but other choices could be made. Using the Condorcet standard, we just need to determine which positional outcomes and tallies allow the “nearly tied” strategic settings when there is a Condorcet winner.

Clearly, a Condorcet winner and loser, respectively, always wins or loses when advanced to a runoff with sincere voting. But, the Condorcet winner need not win with all election procedures; e.g., as shown above, a Condorcet winner can be defeated with an agenda. To simplify the discussion, only three alternatives are considered, but the conclusions hold for any number of candidates. The results are somewhat discouraging.

Select a candidate, even at random. For all positional procedures other than the Borda Count and for any choice of the selected candidate including letting her be the Condorcet loser, there are situations where strategic action could elect the specified candidate. Only with the Borda Count is it impossible for small numbers of strategic voters to elect the Condorcet loser. On the other hand, the Condorcet loser can never be strategically elected with positional runoffs, but any other candidate can. At the other extreme, the only positional runoff which ensures that with small numbers of strategic voters a Condorcet winner always is elected (when one exists) is a three-candidate Borda-runoff.

It is somewhat surprising that a Condorcet winner need not be elected even with a positional runoff. One reason is that a Condorcet winner could receive such a low positional tally that she is not a realistic strategic target. For instance, A and C are, respectively, the Condorcet winner and loser should 2 voters prefer ABC , 51 prefer CAB and 50 prefer BAC . As the plurality outcome is CBA with a 51:50:2 tally, Condorcet winner A 's negligible vote would realistically discourage any strategic action on her behalf. Here, any assertion that a carefully coordinated, massive strategic action could elect A must be viewed with scepticism.

Assume that the pairwise outcomes define the transitive ranking ABC . With the exception of the Borda ranking, the accompanying positional ranking can be any of the 13 ways to rank the three candidates (Saari, 1999), and, for any ranking, there exist profiles involving voters of each type. Indeed, for all non-Borda positional

³ A Condorcet loser is a candidate who loses all pairwise comparisons. As argued in Saari (1999, 2000, 2001), I do not find the Condorcet winner or loser to be adequate measures of the voters' wishes.

methods, there are profiles with the specified pairwise rankings where the positional ranking of BCA has a nearly tied BC tally. The profile can have voters preferring C over B , so the above analysis for positional methods proves that the Condorcet loser C could strategically win in any of these elections. A similar argument shows that there are situations where any candidate could win.

The Borda method, however, must strictly rank the Condorcet winner over the Condorcet loser which makes it impossible to have a BCA or a CAB Borda ranking. The only strategic opportunity to elect C is if the Borda ranking is ACB . But in any three-candidate Borda ranking, the difference between the Borda tallies of the Condorcet winner and loser can never be the smallest Borda vote differential (Saari, 1999). So, our assumptions of a small number of strategic voters and only one nearly tied setting means that only the CB tally is nearly tied. As a change in this outcome does not change the winner, there are no opportunities to strategically elect the Condorcet loser.

A similar analysis holds for runoffs. To strategically deny advancing the Condorcet winner A in a Borda runoff, the Borda ranking must be BAC . The same comment about Borda differentials shows that the pair closest to a tie outcome must be B and A rather than A and C . This precludes strategic action, so the Condorcet winner always wins the runoff.

Since any ranking is possible for all other positional procedures, profiles exist where a strategic action will drop a Condorcet winner from the runoff. For instance, the profile where 12 voters prefer ABC , 11 prefer BCA , 10 prefer CBA , and 11 prefer CAB has the specified ABC pairwise ranking and a CAB plurality outcome with the 21:12:11 tallies which indicate a nearly $A = B$ tie. Since A would beat C in the runoff, CBA voters have the incentive to advance B to the runoff. If two of them do so, then the Condorcet winner A strategically loses with a plurality runoff. (Compare this with assertions in Niou, 2001 based on strategies involving massive numbers of voters.)

“If not, then . . .” and other methods

While Black (1958) embraced the Condorcet winner, he also recognized that this winner need not exist. To address this problem, Black’s method selects the Condorcet winner if one exists; if not, the procedure selects the Borda winner. Thus, Black’s method illustrates what is called here “if not, then . . .” procedures; these approaches select the winner from one procedure if a certain criterion is satisfied, but, if not, it selects the winner from another procedure. The analysis of all of these procedures has much the same flavor, so only Black’s method for three candidates is considered here.

Black’s method

Assume that the pairwise rankings define the transitive ABC outcome. Nothing interesting occurs if A also is the Borda winner (because any strategic behavior would require the unlikely setting of two nearly tied outcomes), so let the Borda

ranking be BAC where the strategic goal is to elect B . There are three choices for a close pairwise election; BC , AB , and AC . Strategically changing the first election has no effect on the outcome as A would remain the Condorcet winner. To strategically reverse a nearly tied AB election, the voter must sincerely prefer A to B . But since reversing this outcome makes B , instead of A , the Condorcet winner, such a strategic vote would be personally damaging; there are no strategic options. The only potentially strategy setting, then, involves a nearly tied AC election where strategically changing this outcome would have the Black winner determined by the Borda ranking. To strategically vote for C over A , the voter must prefer A to C . As the voter prefers the outcome B to A , only voters preferring BAC can be strategic, and they can elect their top ranked candidate. Notice, the strategic voters have moderate views about the sincere and strategic winners as well as about the two candidates in the strategic election. Incidentally, this strategy still applies with an “instant Black method” where voters are not completely sure of the Borda outcome.

The remaining scenario electing A is where the pairwise outcomes define the AB, BC, CA cyclic ranking with a ABC Borda ranking. The sincere outcome is A , so any strategic goal is to elect someone else. Since the analysis of how to strategically vote in the Borda election is described above, consider what happens with the various pairwise votes. If the vote defines a nearly tied BC outcome, then, to strategically vote for C over B , the voter must sincerely prefer BC . As the strategic outcome elects C rather than A as the winner, the voter must prefer CA . Thus, in this setting, the only strategic voter type is BCA ; this voter has moderate views about the particular strategic pairwise election.

This is the only strategic opportunity when the pairwise outcomes define a cycle. To explain, notice that with a nearly tied AB , a strategic change in the outcome would elect B over A . Since a strategic voter must prefer AB , such a vote would be personally counterproductive. Similarly with a nearly tied CA outcome, a reversal would elect A . But A already would win, so this vote is not necessary. A similar argument holds if the pairwise votes define the BA, AC, CB cycle.

With Black’s method, the first stage strategic opportunities depend on whether there is, or is not, a Condorcet winner. With a Condorcet winner, the strategy is to create a cycle so that the Borda winner determines the outcome. The strategic types have the Condorcet winner bottom ranked; the strategic opportunity requires a nearly tied pairwise election involving the Condorcet winner and loser and the strategic voter could elect his top ranked candidate. With no Condorcet winner, the first stage strategy is to avoid allowing the Borda winner determine the outcome by strategically creating a Condorcet winner. The opportunity arises with a nearly tied pairwise election which does not involve the Borda winner; the outcome elects the strategic voter’s second ranked candidate.

It is left to the reader to show that with Black’s method, and any “If not, then . . .” method, a cyclic behavior as described earlier holds with any successful strategic action.

4 Monotonicity

If a winning candidate A wins more converts, this should help, not hurt her. To examine this concern, we state that a procedure is *monotonic* if when candidate A is elected with a profile, and if the profile is changed only by improving A 's standing with certain voters, then A still is elected. Notice, this condition requires each voter's relative ranking of all other candidates to remain the same. By relaxing this restriction, extensions emerge. For instance, another condition may allow a voter to also interchange the two bottom ranked alternatives. As we will see, even minor changes lead to different conclusions.

Positional methods

It is well known that positional methods never violate monotonicity. To recover this result, let the original outcome be ABC . Reversing the BC ranking has no effect on the winner, so consider a nearly tied AB tally; the goal is to understand how monotonicity improvements for A can break this tie. In Table 4, all preferences where A could improve and the associated A - B differential are listed in the first and second columns. The first and second rows lists all rankings where A is not bottom ranked with the associated A - B vote differential. The "NA" matrix entries indicate rankings which are "not applicable" under the monotonicity assumption. For the other matrix entries, the procedure can violate monotonicity if this preference change breaks the tie in B 's favor; i.e., if the A - B vote differential is smaller than in the original profile. The entry "None" means that monotonicity is never violated with any positional method. These matrix entries recover the known result that positional methods satisfy monotonicity. This assertion clearly holds for any number of candidates.

As a promised digression, I now use Table 4 to show that positional methods need not satisfy even slight changes in the definition of monotonicity. For instance, modify monotonicity to allow a situation where if A becomes a voter's top choice, his previous top choice could become a bottom choice. An illustrating example is the 1998 Minnesota gubernatorial elections when Jesse Ventura forces so successfully courted Humphrey votes that in press interviews, some of these voters reported that they ended up with Humphrey bottom ranked. This modified form of monotonicity converts the NA* entry into an admissible change; a change which can hurt A if the

Table 4. Violating monotonicity for positional methods

		ABC	ACB	CAB	BAC
		$w_1 - w_2$	w_1	w_2	$w_2 - w_1$
CAB	w_2	NA*	None	NA	NA
CBA	$-w_2$	NA	None	None	NA
BCA	$-w_1$	None	NA	NA	None
BAC	$w_2 - w_1$	None	NA	NA	NA

new differential is smaller, or if $w_2 > w_1 - w_2$. Thus, for all positional procedures where $2w_2 > w_1$, this improved perception of A could hurt A by electing B .

More generally, consider any variation on monotonicity which keeps or improves A 's ranking for each voter. If the variation also allows changes in the relative rankings of other candidates, it could change the vote differentials for other candidates and allow someone else to be elected. This feature occurs with the Maskin (1984) monotonicity from mechanism design (discussed in Nurmi, 1999, 2002) where, for each voter, A 's position relative to each of the other candidates remains the same or improves; there is no restriction on the relative rankings of the other candidates. To see why no positional procedure satisfies this condition, with the original ABC ranking and a nearly tied AB tally, voters previously preferring CBA changing to either BAC or BCA satisfy Maskin's condition but add to B 's tally allowing B to beat A .

Positional runoffs

With positional runoffs being proposed as reform procedures, it is worth identifying where they violate monotonicity. Let the positional ranking be ABC and let A beat B in the runoff. First assume that A is nearly tied with B in one of these elections. If this is the runoff, a monotonicity improvement in A 's standing only helps her. If this near tie is in the positional election, Table 4 shows that a monotonic change also helps A .

It remains to consider a nearly tied BC . A monotonicity violation can occur only if when C is advanced to the runoff, she would beat A . Thus, as already asserted, cyclic behavior must accompany violations of monotonicity for multistage methods.

The monotonicity consequences of a nearly tied BC are listed in Table 5 where the vote differentials are for $B-C$ tallies. For instance, if A advances in a CBA preference to create ACB , this hurts A (by advancing C to the runoff) only if the new $B-C$ differential of $-w_2$ is less than the original $B-C$ differential of $w_2 - w_1$. This $-w_2 < w_2 - w_1$ inequality, or $w_1 < 2w_2$, identifies all procedures which violate monotonicity as this change would hurt A . Comparing Tables 2 and 3 with Table 5 suggests that while violations of monotonicity can occur, they are not as serious as strategic behavior.

Returning to the moderate vs. extreme theme discussed in strategic voting, notice that in the Table 5 setting, monotonicity is violated only when an improvement

Table 5. Violating monotonicity for runoffs

		ABC	ACB	CAB	BAC
		w_2	$-w_2$	$-w_1$	w_1
CAB	$-w_1$	NA	None	NA	NA
CBA	$w_2 - w_1$	NA	$w_1 < 2w_2$	$w_2 \neq 0$	NA
BCA	$w_1 - w_2$	$2w_2 < w_1$	NA	NA	None
BAC	w_1	$w_1 \neq w_2$	NA	NA	NA

in A 's status makes the status between who is advanced to the runoff more extreme (relative to the positional method). In other words, this same "from moderate to more extreme" variation explains violations both of monotonicity and strategic voting.

The remaining scenario has A second ranked in the positional election but winning the runoff. So, let the positional outcome be BAC where A beats B in the runoff. Again, interchanging the top two positional candidates has no effect on the outcome, so consider a nearly tied AC election which determines who is advanced to the runoff. Can an improved perception of A prevent her from advancing to the runoff? The Table 6 differentials between A - C tallies prove that this never happens. The reason is clear; any change improving A 's status only widens the A - C differential in A 's favor.

Table 6. Never violating monotonicity for runoffs

		ABC	ACB	CAB	BAC
		w_1	$w_1 - w_2$	$w_2 - w_1$	w_2
CAB	$w_2 - w_1$	NA	None	NA	NA
CBA	$-w_1$	NA	None	None	NA
BCA	$-w_2$	None	NA	NA	None
BAC	w_2	None	NA	NA	NA

Positional runoffs can violate monotonicity only when monotonic profile changes affect who is advanced to the runoff. This violation requires the top ranked candidate of the original profile to win the original runoff, a near tie between which other candidate is advanced to the runoff, and the original winner would lose should a different candidate be advanced to the runoff. With three alternatives, the plurality, "vote for two," and Borda procedures each have two profile changes which can violate monotonicity; all other procedures have three.

Multi-stage methods, Black, etc

Not all multi-stage methods violate monotonicity. For instance, if all stages involve pairwise comparisons, such as with agendas or tournaments, monotonicity always helps A as it does not affect any other pairwise comparison. This is a special case of a more general observation.

For a procedure to violate monotonicity, the improved perspective of A must influence the relative positional ranking of some other pair of candidates. This occurs when A 's improved status creates a more extreme status which helps the previously lower ranked candidate of the other pair. Furthermore, the new relative ranking of this pair changes which candidates are compared with A at a later stage.

This comment indicates why positional elections never violate monotonicity. To further illustrate, consider the earlier hypothetical procedure where nine candidates are partitioned into three sets of three where the winner of each set is advanced to a final election of three. Since only one candidate is advanced from each subset, this method never violates monotonicity. After all, in any election where A is the positional winner, monotonicity only enhances her standing; by advancing only one candidate, the relative ranking of the other candidates is immaterial. In those sets where A is not a candidate, monotonicity has no effect on the outcome. On the other hand, slightly changing this approach to advance the two top ranked candidates of at least one set creates a method which does violate monotonicity.

5 “No-show” and other problems

Could a voter be rewarded by neglecting to vote? The above arguments prove that this phenomenon (first noticed by Fishburn and Brams 1983) never occurs with positional methods because a failure to vote deducts points from preferred candidates. On the other hand, intuition gained from the above suggests that this behavior can occur with multi-stage methods; by not voting, a different candidate can be advanced to a next stage. But, is this a serious concern? Our methodology identifies when this behavior can occur with different positional-runoff methods. The table suggests that this problem is not overly serious; expect a voter to be hurt by not voting.

Positional runoffs

For A to win in a positional runoff the positional ranking is either ABC or BAC and A beats B in the runoff. Changing a near tie for the top two positions does not affect the outcome, so such settings never allow the no-show problem. Thus, all abstention difficulties occur with a near tie in the tally which determines who is advanced to the final runoff. For the outcome to change, with the ABC election outcome, C has to beat A in the runoff. In the BAC election, when C rather than A is advanced, either B or C is the winner. All possibilities are listed in Table 7.

Table 7. Neglecting to vote for runoffs where A would win

	ABC	C wins	BAC	B wins	C wins
ABC	w_2	Worse	w_1	Worse	Worse
ACB	$-w_2$	NA	$w_1 - w_2$	Worse	Worse
CAB	$-w_1$	NA	$-w_1 + w_2$	NA	NA
CBA	$-w_1 + w_2$	NA	$-w_1$	NA	NA
BCA	$w_1 - w_2$	2nd Choice if $w_1 \neq w_2$	$-w_2$	NA	NA
BAC	w_1	Worse	w_2	1st if $w_2 \neq 0$	Worse

The first column of Table 7 lists all voter types. The second and fourth columns list, respectively, the $B-C$ and the $A-C$ vote differentials. So, if a voter neglects to vote, these point differentials are deducted from the $B-C$ difference; effectively, they now help C in the positional election. Thus, if this number is positive, neglecting to vote helps C advance to the runoff. The third column shows the consequences of the forgotten vote with a ABC positional outcome; a “NA” means it is not applicable because the missed vote did not change who was advanced to the runoff, “Worse” means that the voter got a worse outcome, and the one entry of “Second place” indicates that for the specified positional methods, if this type of voter neglects to vote, his second ranked, rather than bottom ranked candidate will be elected. Here, the no show problem cannot occur with a “vote for two” runoff procedure.

The fifth and sixth columns show the consequences with a BAC positional outcome; the top of each column specifies the final winner. Again, as long as the plurality vote is not used, only one voter type can prosper by not voting; his negligence could elect his top choice. Here, the no-show problem does not occur with a plurality runoff. Because the no-show problem helps only one type of voter and is neutral or hurts all others, this phenomenon should not be treated as an overly serious concern.

Agenda. To illustrate the no-show problem with the earlier agenda (where C is seeded last), suppose some voters forget to vote. Following the now familiar program, consider the $\{A \text{ beats } B\}$ and $\{C \text{ beats } A\}$ scenario which elects C . If the last election is close to a tie, then the outcome could change if voters preferring C to A forget, or refuse, to vote. But since A would be the new winner, these voters are punished for not voting.

Now suppose the AB outcome of the first election is nearly tied. If voters preferring A to B neglect to vote, allowing B to be the winner, then B is advanced to the final vote. If C wins, there is no change. But, if B wins, then this is a preferred outcome if the neglectful voters prefer B to C .

If the last seeded candidate wins in an agenda, then voters who have this candidate bottom ranked and the winner of the first election top-ranked can help elect their second-ranked candidate by not voting. This occurs only if the first election is nearly a tie, and if the sincere pairwise outcomes form a cycle where the sincere winner loses to the forgetful voter’s second ranked candidate.

6 Designing examples

It remains to show how to construct examples. To illustrate the approach (borrowed from Saari, 2001), a profile is created to illustrate the triple problems of strategic behavior, a violation of monotonicity, and the abstention problem for the instant plurality runoff. According to our approach, this profile must have the plurality outcome ABC where the BC tally is nearly a tie. Furthermore, A must beat B in the runoff, but C would beat A .

The construction of all three-candidate examples (Saari, 1999, 2000, 2001) uses the five following working portions of profiles.

For the construction, start with just two voters having the ABC preferences; the positional and pairwise outcomes reflect this unanimity. For instance, the AB , AC , and BC pairwise votes each have a 2:0 tally.

To create a desired pairwise rankings, notice that the three rankings in lines 1 or 2 of Table 8 have each candidate in each position precisely once, so they have no effect on any positional ranking. They do, however, affect the pairwise outcomes; e.g., the line 1 cyclic configuration creates the AB, BC, CA voting cycle with 2:1 tallies. So, to create a profile where C beats A in a majority vote, add x units of this particular profile configuration to generate the desired CA outcomes. Thus, the new AB tally would be $2 + 2x : x$ while the CA tally would be $2x : x + 2$. Simple algebra proves that as long as $2x > x + 2$ or $x > 2$, the pairwise votes have the desired ranking. For instance, adding $x = 7$ units (which is equivalent to adding 21 new voters) creates the new 23 voter profile where A beats B by 16:7 and C beats A by 14:9.

Table 8. Profile components needed to create examples

1	ABC	BCA	CAB	Cyclic
2	ACB	CBA	BAC	Cyclic
3	ABC	CBA		Reversal
4	ACB	BCA		Reversal
5	CAB	BAC		Reversal

To create the desired plurality outcomes, use the “reversal” profile configurations in lines 3, 4, and 5 of Table 8. Because the two rankings in each line reverse each other, they have no effect on the pairwise rankings. They do, however, change the positional rankings. To make A the plurality winner, add y units of the third line and z units of the fourth; these are the configurations with A top-ranked. With the original two voters, the plurality tallies for A, B , and C now are $2 + y + z, z$, and y . To obtain a desired profile with the ABC outcome, select $z = y + 1$ for any $y \geq 0$; by construction, the BC tallies of z and y differ by only one vote. Illustrating with $y = 3$ and adding these units to the above 23 voter profile, we have a 37 voter profile where 12 voters prefer ABC , 4 prefer ACB , 7 prefer CAB , 3 prefer CBA , and 11 prefer BCA . The plurality outcome is ABC with the 16:11:10 tally and the pairwise outcomes of AB, CA have the respective tallies of 23:14 and 21:16.

This profile has the required features needed to illustrate all of the described difficulties which can occur with the plurality runoff. In a plurality runoff, A and B are advanced and A wins the runoff. All three cases involve BCA voters.

- To illustrate strategic voting, if one BCA voter strategically votes as though preferring CBA , then the new plurality outcome of ACB advances C to the runoff to beat A . The strategic voter has a personally better outcome.

- To illustrate a violation of monotonicity, let two BCA voters change their preferences to ABC . This added support for A hurt B , so the new plurality outcome is ACB – but C beats A in the runoff.
- For no-show, if two BCA voters neglect to vote, then the new plurality outcome is ACB by a 16:10:9 tally, and C beats A in the runoff to reward these neglectful voters with a personally better outcome.

7 Conclusion

An objective of this paper is to create an easily used approach which describes the various unsettling aspects of voting in terms of when, by whom, and how they can occur. In doing so, this simpler methodology identifies underlying intuitive concepts behind these peculiarities; e.g., strategic opportunities or monotonicity violations require adopting, intentionally or unintentionally, a more extreme attitude about the candidates (relative to a procedure) in a manner which changes the outcome. This perspective helps to explore other phenomena.

Could casting a truncated ballot, for instance, help a voter? A natural reaction is that it would not because by not providing full information, those candidates favored by the voter will not receive any votes. But a truncated ballot generates a more extreme setting because the excluded candidates now receive zero points. Since this action can change vote differentials of several candidates, it now is easy to identify settings where this action most surely can help this voter. After all, for any positional method other than the plurality vote, not listing the other candidates can increase the differential between a voter's favored candidate and the others. With runoffs and multistage elimination methods, we now know that anything which can alter the vote differential when candidates are on the brink of who should be advanced can alter the outcome. So, at times this action of truncating a ballot can help the voter, at other times it does not.

As another concern, suppose two groups of voters would elect the same candidate using the same procedure. If they unite, would their combined vote elect this common candidate? While common sense suggests this will happen, we now know that a way to examine this situation is to emphasize near ties. By doing so, we would correctly expect that the full group would always elect the common winner when positional methods are used. But, with runoffs, multi-stage methods, we must hesitate because the stages determine who is advanced. Consequently, we should (correctly) suspect that situations exist where someone else could be elected. To indicate this with Black's method, suppose A wins with the first group as she is the Condorcet winner, and with the second group as the pairwise voting cycle requires the Borda winner A to be selected. When analyzing what happens if the first group has a ABC pairwise outcome with the AB tally nearly tied, we immediately recognize that if the pairwise cycle for the second group has a strong BA vote, then the combined group would elect B , not A , already at the first stage.

An interesting feature accompanies almost all of these results. Namely, the plurality, Borda, and "vote for two" methods usually have fewer matrix entries when describing how these election peculiarities can occur. The reasons are immediate. First, the plurality and "vote for two" methods divide candidates into only the two

groups of “the most desired” and “the least desired” relative to the procedure. All other positional procedures allow more kinds of “moderate” views of two candidates, so they have more matrix entries. Similarly, Borda’s method has the same point differential between a first and second or a second and third ranked candidate, so it has fewer “moderate vs. extreme” settings.

Other results arise by comparing the different tables. For instance, comparing Table 7 with Table 6 shows that monotonicity and no-show have different features. Comparing Table 7 with Tables 2 and 3 indicate that the no-show phenomenon is related to strategic voting in the sense that the no-show is one of the two strategic steps. (To be strategic, a voter is a no-show with the sincere profile and a “new” voter with the strategic preference.)

References

- Black, D.: The theory of committees and elections. Cambridge: Cambridge University Press 1958
- Blair, D.: On the ubiquity of strategic voting opportunities. *International Economics Review* **22**, 649–655 (1981)
- Farquharson, R.: Theory of voting. Yale: Yale University Press 1969
- Fishburn, P.: Monotonicity paradoxes in the theory of voting. *Discrete Applied Mathematics* **4**, 119–134 (1982)
- Fishburn, P., Brams, S.: Paradoxes of preferential voting. *Mathematics Magazine* **56**, 207–214 (1983)
- Fishburn, P., Brams, S.: Manipulability of voting by sincere truncation of preferences. *Public Choice* **44**, 397–410 (1984)
- Gärdenfors, P.: Manipulation of social choice functions. *Journal of Economic Theory* **13**, 217–228 (1976)
- Gibbard, A.: Manipulation of voting schemes; a general result. *Econometrica* **41** 587–601 (1973)
- Kalai, E., Muller, E.: Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures. *Journal of Economic Theory* **16**, 457–469 (1977)
- Kemeny, J.: Mathematics without numbers. *Daedalus* **88**, 571–591 (1959)
- Maskin, E.: The theory of implementation in Nash equilibria. In: Hurwicz, L., Schmeidler, D., Sonnenschein, H.: (eds.) *Social goals and social organization: Essays in memory of Elisha Pazner*. Cambridge: Cambridge University Press 1985
- McKelvey, R., Niemi, R.: A multistate game representation of sophisticated voting for binary procedures. *Journal of Economic Theory* **18**, 1–22 (1978)
- Merlin, V., Saari, D.G.: Copeland method II: Manipulation, monotonicity, and paradoxes. *Journal of Economic Theory* **72**, 148–172 (1997)
- Nanson, E.J.: Methods of election. *Trans Proc Roy Society Victoria* **18**, 197–240 (1882)
- Niou, E.: Strategic voting under plurality and runoff rules. *Journal of Theoretical Politics* **13** (2), 209–227 (2002)
- Nurmi, H.: Voting paradoxes and how to deal with them. Berlin Heidelberg New York: Springer 1999
- Nurmi, H.: Voting procedures under uncertainty. Berlin Heidelberg New York: Springer 2002
- Riker, William H.: *Liberalism against populism*. San Francisco: Freeman 1982
- Saari, D.G.: Susceptibility to manipulation. *Public Choice* **64**, 21–41 (1990)
- Saari, D. G.: *Geometry of voting*. Berlin Heidelberg New York: Springer 1994
- Saari, D. G.: *Basic geometry of voting*. Berlin Heidelberg New York: Springer 1995
- Saari, D.G.: Explaining all three alternative voting outcomes. *Journal of Economic Theory* **87**, 313–355 (1999)
- Saari, D. G.: Mathematical structure of voting paradoxes. 1: Pairwise vote. *Economic Theory* **15** 1–53 (2000)
- Saari, D. G.: Mathematical structure of voting paradoxes. 2: Positional voting. *Economic Theory* **15**, 55–101 (2000)

- Saari, D.G.: Chaotic elections! A mathematician looks at voting. Providence, RI: American Mathematical Society 2001
- Saari, D.G., Merlin, V.: Changes that cause changes. *Social Choice & Welfare* **17**, 691–705 (2000)
- Satterthwaite, M.: Strategyproofness and arrow's conditions: existence and correspondences for voting procedures and social welfare functions. *Journal of Economic Theory* **10**, 187–217 (1975)
- Tangian, A. S.: Unlikelihood of condorcet's paradox in a large society. *Social Choice and Welfare* **17**, 337–365 (2000)
- Van Deemen, A.: The probability of the paradox of voting for weak preference orderings. *Social Choice & Welfare* **16**, 171–182 (1999)