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A comparison of Dodgson's method and Kemeny's rule

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Abstract. In an election without a Condorcet winner, Dodgson's method is designed to find the candidate that is "closest" to being a Condorcet winner. Similarly, if the head-to-head elections among all candidates do not give a complete transitive ranking, then Kemeny's Rule finds the "closest" transitive ranking. This paper uses geometric techniques to compare Dodgson's and Kemeny's notions of closeness and explain how conflict can arise between the two methods.

1 Introduction

In an 1874 paper [1], Charles Dodgson (aka Lewis Carroll) proposed a voting method based on the Condorcet criteria: If there is one candidate that beats every other candidate in head-to-head elections, then that should be the winning candidate. However, not every election has a Condorcet winner. Dodgson proposed that the candidate "closest" to the Condorcet winner should be declared the winner. In 1959, John Kemeny proposed a similar method for elections that require a complete ranking as the outcome by choosing the ranking that is closest to the rankings of the voters [2].

Using the geometric techniques developed by Saari [4, 5], we will compare Dodgson's and Kemeny's notions of closeness. While there are similarities, the difference between requiring a complete transitive ranking and requiring only a Condorcet winner leads to tremendous conflict between the Dodgson winner and the Kemeny Ranking.

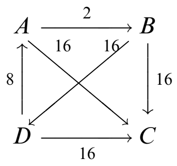
In Sect. 2 we describe Dodgson's method and Kemeny's rule and state our major result. In Sect. 3 we give the necessary geometric framework to compare the methods. In Sect. 4 we explain the geometry of Dodgson's method and relate it to the well-known ℓ_1 metric. Section 5 contains the proofs.

Table 1. An election with 50 voters

Number	Ranking
21	$A \succ B \succ C \succ D$ (1)
12	$C \succ D \succ B \succ A$ (2)
5	$D \succ C \succ A \succ B$ (3)
12	$B \succ D \succ A \succ C$ (4)

Table 2. Head-to-head results from Table 1

	Tally	Margin
$A \succ B$	26,24	2
$B \succ C$	33,17	16
$C \succ D$	33,17	16
$D \succ A$	29,21	8
$A \succ C$	33,17	16
$B \succ D$	33,17	16



2 Dodgson's method

To illustrate Dodgson's Method, consider the election in Table 1 among four candidates, A, B, C, and D with 50 voters, where $A \succ B$ means that A is preferred to B. The head-to-head results from this election are given in Table 2.

Notice that there is no Condorcet winner since the first four head-to-head elections define a cycle where every candidate loses one election. Dodgson's notion is that B is closest to being the Condorcet winner since it loses only one election (to A) by two votes, while A loses to D by eight, C loses to B by 16 and to A by 16, and D loses to C by 16 and to B by 16. Thus, if only two voters with ranking (1) change to $B \succ A \succ C \succ D$, then B will become the Condorcet winner, but any other candidate will require more than two voters to switch their rankings.

2.1 Precise statement of Dodgson's method

In our example, all four candidates are contained in the cycle. However, this may not always be the case as there may be a majority cycle where each candidate in the cycle is preferred to every candidate not in the cycle. In this situation, Dodgson restricts his attention to the majority cycle. We can state Dodgson's Method as follows:

1. If there is a Condorcet winner, then that is the Dodgson winner.
2. If not, there will be a majority cycle. For each candidate in the majority cycle, determine the number of adjacent switches in the voters' preferences that are necessary to make the candidate the Condorcet winner. The candidate in the majority cycle with the fewest required switches is the Dodgson winner.

Table 3. Switches required for the example in Table 1

Candidate	Election lost	Switches	Ranking	Total
A	$D \succ A$	5	(4)	5
B	$A \succ B$	2	(1)	2
C	$B \succ C$	9	(1)	
	$A \succ C$	9	(4)	18
D	$C \succ D$	9	(2)	
	$B \succ D$	9	(4)	18

Applying this to our example, Table 3 shows that B is the Dodgson winner.

2.2 Kemeny's rule

Saari and Merlin [6] give a nice description of Kemeny's Rule that we can apply in our example:

> For each transitive ranking of the candidates, compare the pairwise outcome of the ranking with the pairwise outcome of the profile. Sum the margins of each pairwise outcome where these differ. The transitive ranking with the smallest sum is the Kemeny Ranking.

In our example, the ranking $B \succ A \succ C \succ D$ differs at the $A \succ B$ and $D \succ A$ tallies by margins of 2 and 8, respectively for a total of 10. However, the ranking $A \succ B \succ C \succ D$ only differs at the $D \succ A$ tally by a margin of 8. Notice that any other transitive ranking differs from the profile at one or more of the tallies $B \succ C$, $C \succ D$, $A \succ C$ or $B \succ D$, each of which has a margin of 16. Thus, the Kemeny Ranking is $A \succ B \succ C \succ D$.

This illustrates our main result: The Dodgson winner need not be top ranked by Kemeny's Rule. In fact, we will show the following in Sect. 5.

Theorem 1. *If there are four or more candidates, then there is no connection between the Dodgson winner and the Kemeny Ranking. That is, the Dodgson winner may occur at any position in the Kemeny Ranking.*

3 A geometric framework

In order to understand how Dodgson's method can differ from Kemeny's Rule we need to use the geometric model developed by Saari [4, 5]. Each voter profile specifies the number of voters who prefer each ranking of the candidates. With n candidates $A_1, A_2, \ldots, A_n$, the profile defines a point in $\mathbb{R}^{n!}$ space. For each of the $\binom{n}{2} = \frac{n(n-1)}{2}$ pairwise elections, pick an ordering of the candidates $A_i \succ A_j$ and let a_{ij} denote the margin by which A_i is preferred to A_j in the pairwise vote (if A_j is preferred to A_i, then a_{ij} will be negative). Therefore, the pairwise votes define a point in $\mathbb{R}^{\binom{n}{2}}$ where the sign of any component indicates which candidate won the corresponding election (a zero value indicates a tie).

For example, the profile from Table 1 defines a point in the profile space R^{24} (where 20 of the components are zero) and the corresponding point in the pairwise space $\mathbb{R}^6$ is $(2, 16, -8, 16, 16, 16)$ with the pairwise tallies ordered $(A \succ B, A \succ C, A \succ D, B \succ C, B \succ D, C \succ D)$. Notice that if we have an even number of voters, then each component in pairwise space will be even and if we have an odd number of voters, each component will be odd. Thus, the components of a point in pairwise space corresponding to an actual election will have the same parity.

Further, we gain a linear transformation $\mathbb{R}^{n!} \to \mathbb{R}^{\binom{n}{2}}$ from the profile space to the pairwise space. Saari shows that this is surjective, and as we will show later, every point in pairwise space with integer components of the same parity is the image of a point in profile space with non-negative integer values. In his analysis, Saari normalizes the pairwise tallies to obtain a point in the "representation cube" by dividing each component by the number of voters. However, we will see that Dodgson's method does not respect this scaling. That is, there are profiles that normalize to the same point in the representation cube but have different Dodgson winners.

Note that each orthant in $\mathbb{R}^{\binom{n}{2}}$ determines a ranking, possibly with cycles. In comparing how voting methods based on the pairwise votes treat cycles, the real issue is understanding how each method moves from an orthant representing an intransitive ranking to an orthant representing a transitive one. Saari and Merlin [6] explain that Kemeny's Rule is equivalent to finding the transitive ranking orthant that is the shortest ℓ_1 distance from the pairwise outcome. The ℓ_1 metric, also called the *taxicab* or *Manhatten* metric, determines the distance between two points by summing the absolute value of the differences between the coordinates. For example, the ℓ_1 distance between $(2, -1, 3)$ and $(1, 4, -2)$ is $|2 - 1| + |-1 - 4| + |3 - (-2)| = 11$. Intuitively, we can think of this metric as the shortest driving distance between the points where we are allowed to travel only east-west and/or north-south. We will see that Dodgson's method is closely related to finding the orthant representing the ranking with a Condorcet winner that has the closest ℓ_1 distance to the profile's point in pairwise space.

4 The geometry of Dodgson's method

4.1 Dodgson's method and adjacency switches

A complicating factor in studying Dodgson's method is that it does not depend solely on the pairwise vote but counts the number of adjacent switches required in a voter's ranking. For example, to make A the Condorcet winner in Table 1, we need five voters to switch their $D \succ A$ preference, and we fortunately have 12 voters with the preference $B \succ D \succ A \succ C$ with $D \succ A$ adjacent.

Table 4. A three voter profile

Number	Ranking
1	$E \succ A \succ D \succ B \succ C$ (1)
1	$A \succ C \succ B \succ D \succ E$ (2)
1	$B \succ C \succ D \succ E \succ A$ (3)

Table 5. Head-to-head results from Table 4

	Tally	Margin		Tally	Margin
$A \succ B$	$2 > 1$	1	$A \succ C$	$2 > 1$	1
$B \succ C$	$2 > 1$	1	$A \succ D$	$2 > 1$	1
$C \succ D$	$2 > 1$	1	$B \succ D$	$2 > 1$	1
$D \succ E$	$2 > 1$	1	$B \succ E$	$2 > 1$	1
$E \succ A$	$2 > 1$	1	$C \succ E$	$2 > 1$	1

However, consider the three voter profile given in Table 4 with head-to-head outcomes in Table 5.

The majority cycle contains all five candidates. Notice that both A and B lose just one head-to-head election by a single vote, so if we only viewed the pairwise outcomes, we would expect Dodgson's method to result in a tie. While A requires a single adjacent switch of ranking (1) to $A \succ E \succ D \succ B \succ C$ in order to become the Condorcet winner, B requires two adjacent switches in ranking (2) or (3) to reverse the $A \succ B$ outcome. Therefore, A is the Dodgson winner. Although we cannot simply look at the pairwise tallies to determine the Dodgson winner, the pairwise tallies do give a lower bound for the number of switches that are required for any candidate.

Fortunately for our purposes of comparing procedures, the following lemmas guarantee that there are sufficiently many profiles where all switches occur between adjacent candidates and the Dodgson winner can be determined from the pairwise tallies. We will call a profile with this property a *Dodgson Adjacency Profile*. The following results are proved in Sect. 5.

Lemma 2. *Given any point in pairwise space with all integer entries of the same parity, there is a profile with exactly those pairwise tallies.*

Lemma 3. *Given any point in pairwise space with with all integer entries of the same parity, there is a Dodgson Adjacency Profile with those pairwise tallies.*

4.2 Dodgson's method and the ℓ_1 metric

Intuitively, Dodgson's method appears to be quite similar to Kemeny's Rule: We want to find the ranking (possibly intransitive) with a Condorcet winner that is closest to the pairwise outcome. We will define the ℓ_1 *winner* of a profile to be the Condorcet winner of the ranking corresponding to the orthant that has the smallest ℓ_1 distance to the point in pairwise space among all orthants

Table 6. A profile with 70 voters

Number	Ranking	Number	Ranking
11	$A \succ B \succ C \succ D \succ E$ (1)	14	$B \succ C \succ D \succ E \succ A$ (4)
9	$B \succ A \succ C \succ D \succ E$ (2)	11	$E \succ D \succ B \succ C \succ A$ (5)
14	$E \succ A \succ D \succ C \succ B$ (3)	11	$A \succ C \succ D \succ E \succ B$ (6)

Table 7. Head-to-head results from Table 6

	Tally	Margin		Tally	Margin
$A \succ B$	$36 > 34$	2	$A \succ C$	$45 > 25$	20
$B \succ C$	$45 > 25$	20	$A \succ D$	$45 > 25$	20
$C \succ D$	$45 > 25$	20	$D \succ B$	$36 > 34$	2
$D \succ E$	$45 > 25$	20	$E \succ B$	$36 > 34$	2
$E \succ A$	$39 > 31$	8	$C \succ E$	$45 > 25$	20

that correspond to a ranking with a Condorcet winner. Although Dodgson's Method is closely related to the ℓ_1 winner, there can be differences. The simplest case to see conflict is when the ℓ_1 winner is not in the majority cycle, but there can be other conflicts. Consider the profile in Table 6 with head-to-head outcomes given in Table 7.

Notice that A is the Dodgson winner, requiring five switches in ranking (3) to $A \succ E \succ D \succ C \succ B$, but B is the ℓ_1 winner by a distance of six. The fundamental issue is that there is an added cost in Dodgson's method for each pairwise election that must be changed. For example, there are six switches required to change the three elections which B loses by a total of six votes, whereas the loss of a single election by six votes would require only four switches.

In geometric terms, we can think of this as an added cost in crossing a coordinate plane into another orthant in pairwise space. It is also important to notice that this added cost is the same for each switch, independent of the number of votes that need to be changed. Thus, if the point is farther from a coordinate plane, the effects of this added cost will be reduced. For example, if we simply multiply the number of voters for each ranking by four, our margins increase by a factor of four, and B becomes the Dodgson winner, requiring 15 switches while A now requires 17. This is a striking result: Dodgson's method does not respect scaling. This is the reason why we do not normalize our votes to the representation cube.

This example illustrates the following result which is proved in Sect. 5.

Theorem 4. *Let P be a Dodgson Adjacency Profile for n candidates.*

1. *If $n = 2$ or $n = 3$, then the Dodgson winner is the same as the ℓ_1 winner.*
2. *If $n = 4$ and the ℓ_1 winner is in the majority cycle, then the Dodgson winner and the ℓ_1 winner agree.*

3. *If $n \geq 5$ and the ℓ_1 winner is in the majority cycle, then there is a positive integer N where the ℓ_1 winner and the Dodgson winner agree on any multiple of the profile cP for $c \geq N$.*

This theorem completes the tools we need to compare Dodgson's method with Kemeny's Rule: If we construct a pairwise tally where the ℓ_1 winner is in the majority cycle and conflicts with Kemeny's Rule, then there is a profile that has the same conflict between the Dodgson winner and the Kemeny Rule. Notice that the same idea can be used to compare Dodgson's method to any voting procedure that respects scaling, such as the Borda Count, Copeland's Rule, etc.

The conflict between Dodgson's method and Kemeny's rule occurs because a point in pairwise space may be closer to an orthant representing an intransitive ranking with a Condorcet winner than it is to an orthant representing a complete transitive ranking. As the number of candidates increases, we should expect there to be more conflict between the methods since the number of intransitive rankings with Condorcet winners increases compared to the number of transitive rankings.

5 Proofs

5.1 Proof of Lemma 2

First, suppose that all of the components are even. Consider the two voter profile with the rankings

Number	Ranking
1	$A_1 \succ A_2 \succ A_3 \succ \cdots \succ A_n$
1	$A_n \succ A_{n-1} \succ \cdots \succ A_3 \succ A_1 \succ A_2$

Notice that the pairwise tally is 0 for all pairwise elections except for the A_1, A_2 election where the result is $A_1 \succ A_2$ by a margin of 2. Similarly, we can form profiles that give a complete tie for all pairwise elections except for the A_i, A_j election where $A_i \succ A_j$ by a margin of 2. By taking linear combinations of these two voter profiles with non-negative coefficients, we can construct a profile with the desired outcome.

Now suppose that the point has all odd components. Add to each component the value ± 1 corresponding to the pairwise vote of the ranking $R = A_1 \succ A_2 \succ \cdots \succ A_n$. The new point has all even components, and from above, we know that there is a profile P that has these values as the pairwise outcome. If the ranking R appears in P, then we can remove one voter with ranking R from P to obtain a profile with our original odd pairwise tallies. If R does not appear in P, then we can add to P the profile that has one voter for each of the $n!$ rankings. Notice that this will not affect the pairwise tallies, but we have guaranteed that R appears in our profile. As before, we can remove R from this profile to gain a profile that has the desired pairwise tallies. □

5.2 Proof of Lemma 3

From Lemma 2, we know there is a profile P that has exactly these pairwise tallies. Among all candidates, let t be the largest amount by which any pairwise election $A_i \succ A_j$ fails to have enough adjacent switches in P. Add to P the profile with $t \cdot n!$ voters where each of the $n!$ rankings has t voters. As before, this will not change the pairwise tallies but will guarantee that there are enough voters that have each pair of candidates adjacent that Dodgson's method will require only adjacent switches. □

5.3 Proof of Theorem 4

Throughout Sect. 5.3 we assume that all profiles are Dodgson Adjacency Profiles. In order to prove Theorem 4, we need to understand the relationship between the ℓ_1 metric and the number of switches needed to make a candidate the Condorcet winner. This leads to the following definition.

Definition 5. *The Dodgson distance of a candidate A for a given profile, denoted $Dd(A)$, is the number of switches required to make A the Condorcet winner.*

Similarly, define the ℓ_1 distance of A to be the shortest ℓ_1 distance from the point in pairwise space to a region where A is the Condorcet winner.

Lemma 6. *Suppose that candidate A loses k head-to-head elections by margins of $a_1, a_2, \ldots, a_k$. If there are an odd number of voters, then*

$$Dd(A) = \tfrac{1}{2}(a_1 + \cdots + a_k + k)$$

If there are an even number of voters then

$$Dd(A) = \tfrac{1}{2}(a_1 + \cdots + a_k + 2k)$$

Proof. Suppose there are an odd number of voters. Then each a_i is odd and $\frac{a_i + 1}{2}$ switches are required to make A the winner of the pairwise election. (e.g. If $B \succ A$ by a margin of 9, then five switches are necessary to make A the winner.) Therefore, the total number of switches required is

$$Dd(A) = \frac{a_1 + 1}{2} + \cdots + \frac{a_k + 1}{2} = \frac{1}{2}(a_1 + \cdots + a_k + k)$$

Similarly, if there are an even number of voters, then each a_i is even, and each pairwise election requires $\frac{a_i + 2}{2}$ switches to make A the winner. Simple algebra gives the second claim. □

Note that the extra term of k or $2k$ inside the parentheses is the source of the conflict between Dodgson's method and the ℓ_1 winner. This term explains the intuition of the added cost in Dodgson's method of crossing the coordinate planes.

Lemma 7. *Let P be a profile where A and B are distinct candidates in the majority cycle where $Dd(A) > Dd(B)$ but A has a smaller ℓ_1 distance than B. Further suppose that A requires switching tallies $a_1, \ldots, a_k$ and B requires switching tallies $b_1, \ldots, b_j$.*

1. *If there are an odd number of voters, then*

$$a_1 + \cdots + a_k + (k - j) > b_1 + \cdots + b_j > a_1 + \cdots + a_k$$

If there are an even number of voters, then

$$a_1 + \cdots + a_k + 2(k - j) > b_1 + \cdots + b_j > a_1 + \cdots + a_k$$

2. *There is an integer N where $Dd(A) < Dd(B)$ for any multiple of our profile cP with $c \geq N$.*

Proof. Since A has a smaller ℓ_1 distance than B, we clearly have $b_1 + \cdots + b_j > a_1 + \cdots + a_k$. Suppose there are an odd number of voters. Since B has a smaller Dodgson distance, the previous lemma gives $\frac{1}{2}(a_1 + \cdots + a_k + k) > \frac{1}{2}(b_1 + \cdots + b_j + j)$ and algebra yields the remaining inequality. An identical analysis gives the result for an even number of voters.

To prove part 2, first assume that there are an odd number of voters. Pick N large enough that

$$b_1 + \cdots + b_j > a_1 + \cdots + a_k + \frac{k - j}{N} \tag{5.1}$$

Notice that we can find such an N since the ℓ_1 distance of A is less than the ℓ_1 distance of B and that this inequality holds if we replace N with any $c \geq N$. For any profile cP, A will now lose the pairwise elections by tallies of $ca_1, \ldots, ca_k$ and B will lose by tallies of $cb_1, \ldots, cb_j$. Equation 5.1 gives

$$c(b_1 + \cdots + b_j) > c(a_1 + \cdots + a_k) + (k - j)$$

$$\frac{1}{2}(cb_1 + \cdots + cb_j + j) > \frac{1}{2}(ca_1 + \cdots + ca_k + k)$$

Therefore, $Dd(B) > Dd(A)$ in the profile cP. A similar argument holds for an even number of voters with the $\frac{k - j}{N}$ term replaced with $\frac{2k - 2j}{N}$. □

Notice that part 1 shows that if A is the ℓ_1 winner and B is the Dodgson winner, then B must lose at least two more elections than A.

We are now ready to prove Theorem 1. If $n = 2$ or $n = 3$, then it is impossible for a candidate to lose two or more pairwise elections than another candidate. Thus, the Dodgson winner and ℓ_1 winner must agree. If $n = 4$, then the only conflict can occur if the Dodgson winner loses two or three pairwise elections. In the first case, the ℓ_1 winner must lose no elections making it a Condorcet winner, and in the second case the Dodgson winner would not be in the majority cycle.

If $n \geq 5$, suppose that A is the ℓ_1 winner. Consider each candidate with a smaller Dodgson distance than A, and take the maximum value of N that guarantees A has a smaller Dodgson distance in NP. Then A will be the Dodgson winner in cP for all $c \geq N$. □

5.4 Proof of Theorem 4

We will define profiles where the Kemeny ranking is $A_1 \succ A_2 \succ \cdots \succ A_n$ but the Dodgson winner is A_j, $1 \leq j \leq n$.

By Lemma 3 and Theorem 1, it suffices to find a point in pairwise space with all odd (or all even) entries where the Kemeny ranking is $A_1 \succ A_2 \succ \cdots \succ A_n$ but A_j is the ℓ_1 winner and is in the majority cycle. For $j > 1$, our profiles will include every candidate in the majority cycle and will also include a heavily weighted cycle of all candidates except for A_j to guarantee that the ranking corresponding to the ℓ_1 winner will not be a complete transitive ranking.

If $j = 1$, then any point in the $A_1 \succ A_2 \succ \cdots \succ A_n$ orthant whose entries have the same parity gives the desired Kemeny ranking and A_1 is the Dodgson winner.

Suppose that $1 < j < n$. Begin with a point in pairwise space corresponding to the profile with $4n + 1$ voters all with the same preference $A_1 \succ A_2 \succ \cdots \succ A_n$. Amend the pairwise votes by changing

Pairwise Preference	New Value
$A_i \succ A_j$, $i < j$	1
$A_n \succ A_1$	$2n + 1$

This is illustrated in Fig. 1. Note that all of the pairwise votes are odd and that we have a majority cycle that includes all candidates. Clearly A_j is the ℓ_1 winner by a distance of $j - 1$ since every other candidate loses at least one election by a margin of $2n + 1$. However, the closest transitive ranking is $A_1 \succ A_2 \succ \cdots \succ A_n$ with a distance of $2n + 1$ obtained by changing the $A_n \succ A_1$ election. We can see this since any other attempt to break the cycle would require changing at least one margin of $4n + 1$.

If $j = n$, then we cannot use the same technique since the Kemeny ranking would be $A_n \succ A_1 \succ A_2 \succ \cdots \succ A_{n-1}$. In this case, begin with a point in

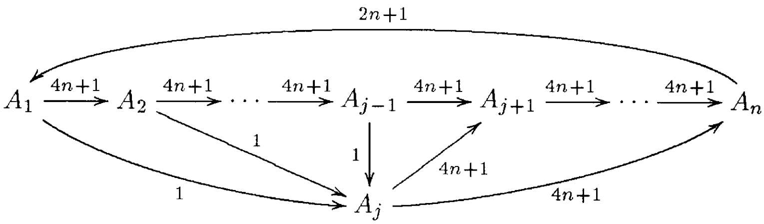


Fig. 1. Profile for $1 < j < n$

pairwise space corresponding to the profile with $6n+1$ voters all with the same preference of $A_1 \succ A_2 \succ \cdots \succ A_n$ and amend as follows:

Pairwise Preference	New Value
$A_i \succ A_n,\ 1 < i < n$	3
$A_n \succ A_1$	1
$A_{n-1} \succ A_1$	$4n+1$

This is illustrated in Fig. 2. As before, all of the pairwise votes are odd and the majority cycle includes all candidates. Notice that A_n is the ℓ_1 winner by a distance of $3(n-2)$ and the Kemeny ranking is $A_1 \succ A_2 \succ \cdots \succ A_n$ by a distance of $4n+2$, obtained by changing the $A_{n-1} \succ A_1$ and $A_n \succ A_1$ elections. □

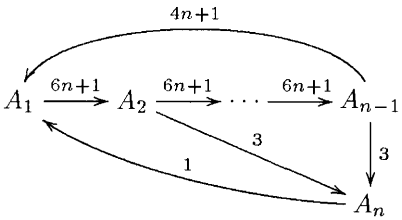


Fig. 2. Profile for $j = n$

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