A PROOFS COURSE THAT TRANSITIONS STUDENTS TO ADVANCED, APPLIED MATHEMATICS COURSES

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INTRODUCTION

At many institutions, a proofs course is the standard to transition undergraduate students from the calculus sequence to upper level courses in mathematics. One of its purposes is to mature undergraduate students and change their perspective from problem solving to theorem proving. In such a course, students learn about the abstract nature of mathematics while learning how to construct basic proofs, how to read mathematics, and how to write mathematics. Of course, it is impossible to teach "how to prove," without proving something! It is often the case that proofs courses introduce concepts and topics from a variety of mathematical fields, providing a sample of advanced pure mathematics.

A survey of some recent textbooks designed for proofs courses indicate the wide variety of topics used to introduce concepts of proof. For example, Schumacher [11], Eisenberg [4], and Fletcher and Patty [5] focus on number theory, axiomatic approaches to examining the real numbers, and the cardinality of sets. Rotman [10] offers less of a sampling of higher mathematics, but grounds the proofs in mathematics more familiar to students, including geometry, trigonometry, and properties of polynomials; of course, the treatment is much more precise and rigorous than the students may have seen and does develop and use more advanced mathematics in these more familiar areas. D'Angelo and West [16] provide a more extensive sampling of advanced mathematics, including discrete mathematics (probability, combinatorics, graph theory, and recurrence relations) and continuous mathematics (sequences, series, continuity, differentiation, and Riemann integration). All of the aforementioned texts have chapters or appendices that introduce elementary set theory, induction, and properties of functions and relations, as well as equivalence relations.

Despite the notion of a proofs course containing a stable of techniques to prove different assertions, not all of the texts include chapters on proof techniques, quantifiers, logic, *etc.* Still there are other texts that focus on the processes of proving and writing mathematical results. These include Solow [14] and Velleman [15]. In particular, Velleman [15] breaks the process of proving a result into smaller pieces and discusses how scratch-work evolves into the final wording of a proof.

There are a number of takes on how to transition students to upper level mathematics. Like the aforementioned texts, our proofs course introduces students with a calculus background to standard techniques of proof through different topics in mathematics. Motivated by our attendance at an NSF workshop on Project InterMath, we decided that using the transition from discrete to continuous mathematics would provide a good setting for a proofs course at the sophomore level. The mathematical topics in the course come from difference equations, differential equations, and elementary linear algebra. However, rather than consider the course as a sampler of advanced pure mathematics courses, we view our course as a sampler of the many applications of mathematics.

For our institution, we believe that not only should the proofs course enhance the ability of our students to communicate verbally and through writing, but it should also contain a heavy technological component and exploratory aspect. We want students to be able to make conjectures and have an experience that introduces them to the process of research. As for the mathematical content of the course, we considered what materials we wanted our students to know *before* they entered our upper level courses. In this article, we not only describe the course, but also give the background of our institution and how this course was designed to solve certain problems in our program. By design, the proofs course is to give a sampling of the different majors and programs offered in our department. Although the course will run as an elective in academic year 2003-2004 and will become part of the required curriculum in 2004-2005, much of the content for the course has been used by the authors in courses in Calculus I/Discrete Dynamical Systems, Game Theory, and Linear Algebra at the U.S. Military Academy at West Point, Game Theory, Differential Equations, and Mathematical Modeling at Montclair, and Differential Equations at Rutgers. We include an outline of the course and examples of some of the content from the course. We conclude by discussing how to adapt our course to another institution, as well as ruminate on the purpose of a proofs course with suggestions on how to develop a course tailored to other constraints.

FITTING THE PROOFS COURSE INTO THE INSTITUTION

Montclair State University's Department of Mathematical Sciences graduates about 30 majors in mathematics every year. Approximately 35% of our majors are transfer students from other institutions, primarily from the local community or county colleges of New Jersey. A number of our majors begin by taking remedial courses in mathematics—typically one semester of Pre–Calculus.

Table 1 gives an outline of the typical mathematics courses our majors take. We have two mathematics streams—Mathematics and Applied Mathematics—the latter being further sub–divided according to specialization into Track I (Discrete Applied Mathematics and Operations Research) and Track II (Statistics). The minimum required credits in mathematics for a typical student majoring in mathematics is 40 while a student majoring in the applied mathematics concentration takes 49; c.f. Table 1. The total credit requirements for our majors is specified in Table 2. Additional mathematics courses are chosen as *free elective* (see last row of Table 2) requirements by the students. The only courses that all majors take at the freshman/sophomore level are the Calculus sequence and Linear Algebra. Many students take the Calculus courses off sequence (beginning Calculus I in the spring of their freshman year) because they take Pre-Calculus in the fall of their freshman year. These students may take Calculus III and Linear Algebra concurrently to "catch up" in the spring of their sophomore years. Despite the emphasis to take these courses by the end of the sophomore year, some students take Linear Algebra in the fall of their junior year while also taking more advanced courses. Montclair State University's Calculus sequence is fairly traditional with nods to calculus reform. All Calculus courses use the Larson, Hostetler, and Edwards [8] text. The department policy is for all students to have a TI-86 calculator; students are allowed to use

<table>
<tr><td colspan="3">All Majors (19 credits)
Calculus I, II, III + Linear Algebra + Probability</td></tr>
<tr><td rowspan="3">Mathematics
(9 credits)
Foundations of
Computer Science I
Advanced Calculus I
Foundations of Modern Algebra</td><td colspan="2">Applied Mathematics (9 credits)
Foundations of Computer Science I + II
Introduction to Mathematical Modeling</td></tr>
<tr><td colspan="2">Specialization</td></tr>
<tr><td>Track I (9 credits)
Discrete Math
Operations Research I
Operations Research II</td><td>Track II (9 credits)
Statistical Methods
Statistical Computing
Mathematical Statistics</td></tr>
<tr><td>At least 4 electives
(12 credits)</td><td>At least 4 electives
(12 credits)</td><td>At least 4 electives
(12 credits)</td></tr>
<tr><td colspan="3">Major requirements (Mathematics courses)</td></tr>
<tr><td>(19 + 9 + 12) = 40 credits</td><td colspan="2">(19 + 9 + 9 + 12) = 49 credits</td></tr>
</table>

Table 1: Required and elective (mathematics) courses for Mathematics and Applied Mathematics Majors at Montclair State University

their calculator in lectures, on quizzes, and for exams. It is expected that faculty will incorporate the graphing calculator into their lectures. Because the department does not require group work or written assignments in the Calculus sequence, student exposure to these pedagogical devices is largely instructor dependent. It is possible for students in the higher–level courses to have no experience with group work, exploration, or writing.

From this point onward, we use the phrase *higher–level courses* to refer to any course beyond the required courses of the Calculus sequence (Calculus I–III), Linear Algebra, and Probability. Because some higher–level courses are offered every term while some are offered every year, and do not typically have pre-requisites other than some of the five aforementioned courses, the background of students in the higher–level courses can vary dramatically. There are students in the last term in their undergraduate careers, as well as students who are taking their first higher–level courses. The proofs course decreases the degree of variability of mathematical preparation of students in higher–level courses.

Some of the key points about the knowledge, preparedness, and maturity of our students are listed below:

- Almost all our students taking higher–level courses have had no exposure to proofs (except in their Linear Algebra class). Due to the way our required courses are structured, students typically do not have a chance to get abundant practice with descriptive writing projects.

- Our students have little or no experience with computer algebra systems (for example Maple) or spreadsheets (for example Excel).

- The knowledge base of our students when they begin taking higher–level courses is limited; for example, they do not have any exposure to discrete dynamical systems or differential equations (the only exception being brief reference to separable equations in calculus). Moreover, they do not get a flavor for the many possible applications of mathematics.

The proofs course is designed to address most of the issues addressed above. Before the introduction of the proofs course, any student with the pre–requisites of the calculus sequence and linear algebra could take any of the higher–level courses. Hence, the mathematical maturity and experience of students in these courses varied greatly. The introduction of a new sophomore level pre–requisite increases the minimal experience of the students in the higher–level courses. Also, since a number of our transfer students take linear algebra at other institutions, the proofs course ensures that students in the higher–level courses have been introduced to proofs, applications, and exploration.

Students in the proofs course are introduced to the rudiments of writing proofs early in the term. This is achieved by introducing proofs in the context of difference equations; for example, using induction to prove the form of a general solution to a second order, linear homogeneous difference equation. The student is encouraged to explore the structure of the general solution using Maple or Excel, conjecture the form of the solution, and then prove their conjecture using induction. The close interplay between exploration, conjecture, and proof forms a structure which is reinforced throughout the course as new topics are introduced.

Since Montclair does not have a sophomore level differential equations course, some students may graduate from our program with only the limited exposure to differential equations that they received in the calculus sequence. The proofs course gives the students a better idea of the usefulness of differential equations, as well as the elementary solution techniques. Our course is designed to introduce students to applications of mathematics to "real world" problems at an early stage using difference and differential equations as tools for modeling. Most of our mathematics majors intend on becoming high school mathematics teachers. Montclair has a strong Mathematics Education group in the department and it is competitive to enter the teacher education program majoring in mathematics. Many students decide to major in mathematics, intend to teach, but are not interested in the teacher education program. It is our belief that some of these students decide that they want to teach because they like mathematics, but are unaware of other career opportunities.

By introducing a variety of applications, the proofs course demonstrates that mathematics is a tool that can be used to solve and model problems in different fields. The proofs course also provides an opportunity to introduce students to the areas of applied mathematics in which the faculty are currently active. By selecting applications that are germane to the research of the faculty, the students get to know what the faculty are doing and learn about opportunities in undergraduate research. A handful of faculty at Montclair are involving undergraduates in their research, but the common complaint is that the student becomes involved in the research too late in his or her undergraduate career. A detailed description of the course is given in the next section.

OUTLINE, DESCRIPTION, AND PHILOSOPHY

The pre–requisite for the course is two semesters of calculus. The course material emphasizes exploration, applications, technology, and proofs. The course combines topics from Discrete Dy-

Mathematics major	Applied Mathematics major
Major requirements (mathematics courses) from Table 1	
40 credits	49 credits
Collateral course requirements (non–mathematics courses)	
8 credits	7–9 credits
General Education requirements (non–mathematics courses)	
46 credits	46–48 credits
Free elective requirements (mathematics (or other) courses)	
26 credits	14–18 credits

Table 2: Mathematics, Collateral, General Education, and Free elective requirements for Mathematics and Applied Mathematics majors at Montclair State University

namical Systems, Linear Algebra, and Differential Equations with a strong exploratory and writing component. Students are required to work in groups and turn in projects throughout the term, culminating in a capstone project at the end of the course. Many of the applications are taken from Interdisciplinary Lively Application Projects (ILAPs) by Arney [1]. ILAPs originated at the United States Military Academy at West Point through the interaction of the mathematics department with the science, engineering, and social science departments. ILAPs are integrated, student-centered projects linking mathematics with partner disciplines.

In addition to the projects, students are required to conjecture results relating to the topics being covered through exploration and write detailed and comprehensive proofs individually. Approximately half the class time is spent on exploration while the other half is dedicated to rudimentary proof techniques. Our philosophy of using exploration as a preamble to mathematical rigor and proof has been explored and used in proofs courses before. During the 1990's, Mount Holyoke [9] developed a "bridge" course titled *Laboratory in Mathematical Experimentation.* Similar to our proposed transitions course, this course is taken by all mathematics majors at the beginning of their sophomore year; the pre–requisite is two semesters of Calculus. The course at Mount Holyoke lets students learn a wide range of topics in mathematics through discovery and experimentation. The students, working in small groups, are encouraged to explore topics, make conjectures and construct arguments in support of those conjectures. The preface in [9] states that

> Students who have taken the "the Lab" course are more likely to ask questions and look for patterns, to formulate arguments clearly, and more likely to dive in and "mess around" with a hard problem. Moreover, students who have taken the course do better in Real Analysis and Abstract Algebra than students who have not.

The Mount Holyoke course focuses on proofs and pure mathematics through exploration. The spirit of our course is very similar to the Mount Holyoke course, but it is centered on proof and exploration in applied mathematics.

The ILAP's form a collection of engaging problems which excite the students. Our course is designed to use the students' curiosity about these applications as a starting point to help them learn the value of exploration, to guide them into forming conjectures from their explorations, and to teach them how to use mathematical reasoning to prove their conjectures. D'Angelo and West [16] mention that the inherent difference between the focus on computation in lower–level courses and on attention to careful exposition in the higher–level courses is a major challenge to students. Various proofs and transition courses have different approaches to bridge this gap. These approaches are usually based on specific needs and focus of the institutions where the courses are introduced. The lack of a required differential equations course at Montclair State University and the attempt to attract more students who will be interested in applications of mathematics led us to our take on the transition course. An outline of the specific topics covered in the course appear below:

- **Weeks 1–2:** Review sequences from Calculus II. Introduce Discrete Dynamical Systems. Examine first-order, linear homogeneous difference equations. Explore numerical solutions and their long-term behavior using Excel. Introduce a detailed application through an ILAP (for example, saving to buy a car).
- **Weeks 3–4:** Conjecture solutions and introduce induction to prove conjectures. Analytically solve difference equations. Continue to apply difference equations to model real-world behavior. Assign first project.

A specific example on the use of induction to prove the general form of the solution to a second order, linear homogeneous difference equation is given in the next section. The example illustrates the *exploration leading to conjecture leading to proof* structure of our course. During Weeks 2–3 the students explore specific examples of difference equations. In particular, they use spreadsheets (for example Excel) to study the long term behavior of solutions and to conjecture the structure of general solutions. Finally, they prove their conjecture rigorously. The proof is done carefully in class for at least one case (for example, when the characteristic equation of a second order linear homogeneous difference equation has distinct roots), and the other cases are assigned are homework.

- **Weeks 5–7:** Consider systems of difference equations by introducing elementary ideas from Linear Algebra. Have students explore ideas about stability, eigenvalues, and eigenvectors using Maple. Develop models to highlight concepts and applications.
- **Weeks 8–9:** Use limits to transition from difference equations to differential equations. Examine slope fields with Maple. Assign second project.

Develop models to highlight concepts. A specific example from evolutionary game theory is given in the next section. The mating strategies used by lizards in California is modeled using discrete and continuous models. This provides an opportunity to illustrate the transition and connections between difference equations, which is the main topic in the first third of the course, and differential equations, which forms the last third.

- **Weeks 10–12:** Consider first and second order differential equations with constant coefficients. Analyze a real-world application using Maple (for example, population models). Assign capstone project.

- **Weeks 12–14:** Examine systems of linear differential equations including simple concepts of stability. Compare discrete and continuous models using technology. Develop models of real-world phenomena (for example, predator–prey models).

- **Week 15:** Small groups report on capstone projects.

As an illustration of the interplay between exploration, conjecture, and proof is the material on fundamental solutions of linear homogeneous differential equations in weeks 12–14. The material is standard and can be found in standard texts on differential equations, for example, Boyce and DiPrima [2].

Let us consider the simple second order homogeneous equation $y'' - y = 0$ where $y = y(t)$ is the solution. Building on their knowledge of the exponential function from Calculus, the students quickly verify that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions. Further exploration leads to the fact that any function in the family $y(t) = c_1 y_1(t) + c_2 y_2(t)$, where c_1 and c_2 are arbitrary constants, is a solution. Computer algebra systems such as Maple are used to visualize the solution family for a range of c_1, c_2 values. As a next step the students explore and discover that specific solutions can be identified in the family by asking questions like "can we identify the solution which passes through he point $(0, 4)$ and has a slope -2 at that point"?

The explorations parallel theorems which demonstrate the steps needed to make the transition from computation to mathematical rigor. In what follows we assume the existence and uniqueness result for the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = 0 \text{ with } y(t_0) = y_0 \text{ and } y'(t_0) = y'_0,$$

where $p(t)$ and $q(t)$ are continuous functions for $\alpha < t < \beta$. We do not delve into the details of this involved existence theorem. However, there are examples of Theorems which parallel the explorations and use simple direct proof techniques. For example

Theorem 1 (Principle of Superposition). *If y_1 and y_2 are two solutions of the differential equation $L[y] = 0$, then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants c_1 and c_2.*

The question *can we find c_1 and c_2 so as to satisfy the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$*, leads naturally to the definition of a Wronskian and the theorem

Theorem 2. *Suppose that y_1 and y_2 are two solutions of $L[y] = 0$, and that the Wronskian $W(y_1, y_2) = y_1 y'_2 - y'_1 y_2$ is not zero at the point t_0 where the initial conditions are assigned. Then there is a choice of constants c_1, c_2 for which $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfies the differential equation $L[y] = 0$ and the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$.*

Finally, the students are led to the theorem characterizing the general solution

Theorem 3. *If y_1 and y_2 are solutions of the differential equation $L[y] = 0$, and if there is a point t_0 where the Wronskian $W(y_1, y_2)(t_0)$ is nonzero, then the family of solutions $y = c_1 y_1 + c_2 y_2$ with arbitrary constants c_1 and c_2 includes every solution of $L[y] = 0$.*

Proving the last theorem requires more sophistication and maturity than the simple, direct proofs of the previous two. In particular, the students see how the uniqueness of solutions plays a fundamental part in the proof.

SOME SPECIFIC MATHEMATICAL CONTENT

Mathematical induction is often the first type of proof that is taught to students. Students in the proofs course use induction to prove that solutions of a particular form are general solutions to linear, homogeneous difference equations. To demonstrate the use of induction, we provide the following example.

Example 1. *Use of induction to prove the form of a general solution to a second order, linear homogeneous difference equation.*

Assume that the second order, linear homogeneous difference equation is of the form $a(n) = sa(n-1) + ta(n-2)$ where s and t are constants and a is a function of n that satisfies the given recursive relationship. The characteristic polynomial associated with the difference equation is $x^2 = sx + t$.

Suppose that the characteristic polynomial has distinct roots r_1 and r_2. Then, the general solution to the difference equation has the form $a(n) = c_1 r_1^n + c_2 r_2^n$ where c_1 and c_2 are arbitrary constants that are determined by the initial conditions. To prove this by induction, assume that $a(n) = c_1 r_1^n + c_2 r_2^n$ for non-negative integers less than or equal to n. By substitution, it follows that

$$\begin{aligned} a(n+1) &= sa(n) + ta(n-1) \\ &= s\left(c_1 r_1^n + c_2 r_2^n\right) + t\left(c_1 r_1^{n-1} + c_2 r_2^{n-1}\right) \\ &= c_1\left(s r_1^n + t r_1^{n-1}\right) + c_2\left(s r_2^n + t r_2^{n-1}\right) \\ &= c_1 r_1^{n+1} + c_2 r_2^{n+1}, \end{aligned}$$

where this last equality follows by substitution since r_1 and r_2 are the roots of the characteristic polynomial. Hence, by induction, if the roots are distinct, then the general solution is $a(n) = c_1 r_1^n + c_2 r_2^n$.

Suppose that the characteristic polynomial has a repeated root, r. Then, the general solution to the difference equation has the form $a(n) = c_1 r^n + c_2 r^n n$ where c_1 and c_2 are again arbitrary constants that are determined by the initial conditions. To prove this by induction, assume that $a(n) = c_1 r^n + c_2 r^n n$ holds for non-negative integers less than or equal to n. Since the root is repeated, the characteristic polynomial is $(x-r)^2 = x^2 - sx - t$. This implies that $t = -r^2$ and

$s = 2r$. By multiple substitutions, it follows that

$$\begin{aligned}a(n+1) &= sa(n) + ta(n-1) \\ &= s\left(c_1 r^n + c_2 r^n n\right) + t\left[c_1 r^{n-1} + c_2 r^{n-1}(n-1)\right] \\ &= c_1\left(sr^n + tr^{n-1}\right) + c_2\left[sr^n n + tr^{n-1}(n-1)\right] \\ &= c_1\left(2r^{n+1} - r^{n+1}\right) + c_2\left[2r^{n+1}n - r^{n+1}(n-1)\right] \\ &= c_1 r^{n+1} + c_2 r^{n+1}(n+1).\end{aligned}$$

Hence, by induction, if the roots are repeated, then the general solution is $a(n) = c_1 r^n + c_2 r^n n$.

We consider two formulations of the same biological model of the competitive mating strategies of lizards in California. This example was motivated from an article in *The Economist* [7] that reported on an article in *Nature* by Sinervo and Lively [13]. A more complete comparison of the differences between the discrete and continuous approaches appears in Weibull [17] and has been successfully used in game theory courses by Jones to focus on the subtleties of modeling dynamic behavior. The example demonstrates the applicability of mathematics to model reality, but also introduces students to areas of research (mathematical biology and game theory) in which Montclair faculty are active. A short discussion on implementation and pedagogy follow the example.

Example 2. *An application to evolutionary game theory/biology that depends on the discrete or continuous formulation.*

Rock-Paper-Scissors or Roshambo is a two-player game with no clear choice of which of the three options is best to employ. Rock crushes scissors, paper covers rock, and scissors cuts paper. The play results in a tie if both of the players select the same action, *e.g.*, rock versus rock. If players in a population only play one of the three alternatives and are randomly matched with an opponent, then the optimal play would be to play the strategy that defeats the most frequently played action. Without such frequency information, it is not surprising that the best strategy is to randomize between rock, paper, and scissors, playing each with probability $\frac{1}{3}$. What is surprising is that Rock-Scissors-Paper models the mating strategies of lizards in California. As reported in Sinervo and Lively [13], male lizards with different colored throats use different mating strategies and pass their strategies to their similarly colored male offspring. The relationship between the strategies employed by the lizards is the same as the relationship between the strategies of playing rock, paper, or scissors in Roshambo. That is, there are three strategies and each strategy becomes more successful when another strategy becomes more frequent. For a more detailed explanation of the strategies, placing them into the context of the actual practices of the lizards, consider Sinervo and Lively [13] or the website [12].

When a mating strategy becomes more successful, it results in more offspring of lizards that employ the strategy. The evolutionary process can be modeled both discretely and continuously. We consider the discrete process first. We revert back to Rock-Paper-Scissors to discuss the evolutionary process. Let (a, b, c) where $a+b+c = 1$ and a, b, and $c \geq 0$ represent the percent of the population at time n playing "rock," "papers," and "scissors." The generation of the population at time $t+1$ depends on the population distribution at n. How the population evolves depends on the relationship between a, b, and c. Specifically, the "rock" players tie against one another, lose against "paper" players, and win against "scissors" players; this occurs with probability a, b, and c respectively. A player receives 2 points for a win, 1 point for a tie, and 0 points for a loss. The distribution of players at time $n+1$ is the percentage of points that each strategy receives. "Rock," "paper," and "scissors" players receive $a+2c$, $b+2a$, and $2b+c$ points respectively. Since the sum of all points is $a+2c+b+2a+2b+c = 3(a+b+c) = 3$, the distribution of the population at time $n+1$ is $\left(\frac{a+2c}{3}, \frac{b+2a}{3}, \frac{2b+c}{3}\right)$.

Hence, the evolutionary process can be viewed as a Markov chain or system of difference equations where the population at time t is given by $[a(n)\ \ b(n)\ \ c(n)]^T$ and the next generation can be determined by matrix multiplication:

$$\begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a(n) \\ b(n) \\ c(n) \end{bmatrix} = \begin{bmatrix} a(n+1) \\ b(n+1) \\ c(n+1) \end{bmatrix}.$$

As long as the initial population consists of nonzero populations, then the distribution tends to $\left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]^T$. The matrix describing the evolution of the population is doubly stochastic, because all row and columns sum to 1, see, *e.g.*, Isaacson and Madsen [6]. All doubly stochastic matrices of dimension m have the m-dimensional vector with all entries $\frac{1}{m}$ as a fixed point or equilibrium vector. Students can arrive at the equilibrium vector by iterating the matrix. Students can plot the long-term behavior using Excel or Maple and see that the vectors converge to $\left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]^T$.

The above depiction of evolution in a society of rock-paper-scissors players is by finite replicator dynamics, as described in Weibull [17]. However, the evolution can be modeled continuously. Let the current population be given by $[a\ \ b\ \ c]^T$; here, a, b, and c are functions of time. For $s \in \{a, b, c\}$, the growth rate $\frac{s'}{s}$ of the portion of the population using strategy s equals the difference between the strategy's current payoff and the current average payoff in the population. The "rock" strategy's current payoff is given by $a+2c$; the "paper" strategy's current payoff is $b+2a$; the "scissors" strategy's current payoff is $2b+c$. These are calculated as before. The current population $[a\ \ b\ \ c]^T$ receives on average a payoff of

$$a(a+2c) + b(b+2a) + c(2b+c) = a^2 + 2ac + b^2 + 2ab + 2bc + c^2 = (a+b+c)^2 = 1.$$

A little algebraic manipulation and $\frac{a'}{a} = a+2c-1$ or $a' = (a+2c-1)a$. Similarly, for b and c, and the system of differential equations becomes

$$\begin{aligned} a' &= (a+2c-1)a \\ b' &= (b+2a-1)b \\ c' &= (c+2b-1)c. \end{aligned}$$

Once again, the vector $\left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]^T$ is an equilibrium or fixed point. However, unlike in the discrete version, where populations converged to the fixed point, in the continuous model, the vector is not attracting, nor repelling. Indeed, it is a center and trajectories of $[a(t) \;\; b(t) \;\; c(t)]^T$ orbit around the equilibrium. Using Maple, students can plot the trajectory of the orbits on the simplex $S = \{(a, b, c) \mid a + b + c = 1; a, b, c \geq 0\}$.

Pedagogically, the discrete version is introduced earlier in the term (see Weeks 5-7 in the outline). The data presented in Sinervo and Lively [13] and on the website [12] shows oscillatory behavior and the population does not converge to the equilibrium vector, as predicted by the discrete time model. This provides an opportunity to discuss alternate ways to model the changes in the population. The differential equation version of the lizard mating game is considered later in the term (see Weeks 12-14 in the outline). By revisiting the evolutionary model of the lizard population, students learn that there is no one way to model reality. This is also a good time to discuss how a model should be developed and compared to reality. Similarly, the comparisons between the discrete and continuous models should focus on the increased complexity of the continuous model and how simple models are valued, as long as they accurately model reality.

MODIFYING THIS COURSE TO OTHER INSTITUTIONS

To directly apply our twist on a proofs course to another institution seems to require the conditions present at Montclair. In particular, the program should not offer a sophomore-level differential equations course and the program or department should be centered on applied mathematics. Of course, picking specific applications to match the research interests of the faculty depends on the faculty. The applications can be changed according to the program or the individual faculty. However, the transition course could be used *in lieu* of a sophomore-level differential equations course, if the department also offered an upper level course in ordinary differential equations. The students would receive some of the content taught in a typical sophomore-level differential equations course, while gaining a broader perspective about what mathematics is about.

For institutions with a more pure mathematics bent, the spirit of the proofs course can still be implemented with topics that coincide with the faculty's and program's interests. This same benefit of having the students exposed to faculty research areas can increase the likelihood of students being ready to pursue research at an earlier point in their educational career. The exploratory aspect and use of technology can still be employed for a proofs course that surveys advanced pure mathematics; indeed, there are software packages to demonstrate abstract algebra (*e.g.*, the GAP software) and offer students an opportunity to explore and conjecture proofs.

CONCLUSION

The liberating idea from this article is that a proofs course can be tailored to fit the nuances of a specific program and does not have to be a survey of pure mathematics or an introduction to logic. Teaching students about how to read and write proofs can be accomplished regardless of the mathematical content. As opposed to other more advanced courses, proofs courses are less motivated by content and more concerned with providing an environment where students can explore and make conjectures and see how mathematics is done, as opposed to learned. We chose to highlight applied mathematics to coincide with faculty interests and the realization that few of our students go on to graduate school. If the population of students at Montclair changes, our proofs course is easily adapted to respond to such changes.

References

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