Integrating Combinatorics, Geometry, and Probability through the Shapley-Shubik Power Index

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1 Introduction

It is our belief that students compartmentalize mathematical techniques to be used solely for a specific problem or narrow set of problems. Ideally, students would develop a toolbox of mathematical techniques to analyze *any* problem from any of a multitude of perspectives. Analyzing simple weighted-voting games helps students develop varied approaches to problem solving, while demonstrating how to use different mathematical skills in nontrivial, relevant ways. Such an analysis necessitates the integration of mathematical topics, including combinatorics, geometry, and probability.

This article serves as a primer for instructors so that they may introduce simple weighted-voting games and the Shapley-Shubik power index and relate voting theory to various topics in the curriculum. To encourage implementation and adaptation of this material, we include many examples and exercises. For this reason, this article may be used for self study by independent study students. These materials have been developed in a handful of courses at Montclair State University from spring 1999 to the present, including a general education requirement course, an upper level applied combinatorics and graph theory course, a graduate level course in combinatorial mathematics, and two independent study courses. Indeed, one student has used this article as a self-study guide as a precursor to computational voting theory. The different levels of use are a testament to the diversity of mathematics that can be used to analyze simple weighted-voting games through the Shapley-Shubik power index. In Section 6, we provide specific guidelines and suggestions on how to use the material developed in this article for different courses.

Simple weighted-voting games require only basic notions of sets, addition, and inequalities. The Shapley-Shubik power index is defined in terms of permutations of voters. Although both ideas are easy to understand, simple weighted-voting games and their Shapley-Shubik power indices can be rigorously analyzed to lead to interesting mathematics. The following is a sample of the types of mathematics and skills that are needed to analyze simple weighted-voting games and their Shapley-Shubik power indices; it also provides a rough outline of the order that these topics appear in this paper. The analysis of Shapley-Shubik power indices of discrete, simple weighted-voting games requires concrete ideas about domain and range, applications of logic, properties of symmetry, permutations, and the addition of real numbers, as well as the calculation of the number of nonnegative integer solutions of equations and equations with inequality constraints. The analysis of continuous, simple weighted-voting games requires use of equivalence relations and partitions, the geometric relationship between inequalities and half-planes, and how the area of a region yields information about the likelihood that certain outcomes occur. Properties of discrete and continuous games are connected by limits of combinatorial identities that converge to areas of partitions.

We are not the first to suggest using simple weighted-voting games in the undergraduate curriculum. There are some exemplary materials that introduce simple weighted-voting games and the Shapley-Shubik power index in the context of modeling political interactions (*e.g.*, Lampert, 1988, Straffin, 1984, and Straffin, 1979), suitable for discrete math and modeling courses. For this reason, we do not stress the modeling component in this work, although it is a component of our class presentation of this material. We believe that our approach asks more mathematical questions, especially geometrical questions, of the students, while retaining its relevance to political science and modeling. There are other geometric approaches to mathematical political science that are accessible to undergraduate students, including Saari (1994) and Saari (1995). These texts do not focus on simple weighted-voting games, but on election procedures.

A benefit of introducing this material to students is that they can read research on the evolution and formation of political institutions. In particular, there have been accessible analyses of the power indices of simple weighted-voting games modeling the European Economic Community (Brams and Affuso, 1976), European Union (Hosli, 1993, Berg, 1999, and Nurmi and Meskanen, 1999), International Monetary Fund (Dreyer and Schotter, 1980), and the United States' Electoral College (Mann and Shapley, 1964). At the heart of these articles are mathematical phenomena related to the institution at hand.

2 Simple Weighted-Voting Games and the Shapley-Shubik Power Index

Stockholders of a company are often allowed to vote "for" or "against" a potential company policy at a shareholders' meeting. Committee members often vote "yes" or "no" to arrive at a joint decision. Jurors vote "guilty" or "not guilty." All of these situations can be modeled by simple weighted-voting games. In the case of jurors, their votes are treated the same way. However, at a shareholders' meeting, someone who owns more shares of stock of the company has her vote count more; indeed, the vote counts for as many shares of stock the stockholder has. Simple weighted-voting games can model these diverse situations, where voters' votes may be weighted differently. However, simple weighted-voting games only model elections where two outcomes are possible: "yes" and "no." A measure is passed if enough voters vote "yes."

Definition 1. A simple weighted-voting game is a set of n voters $\{v_1, v_2, \ldots, v_n\}$, where voter i's vote carries the weight w_i, and a quota, a value that if the sum of the "yes" voters' weights is greater than or equal to the quota, q, then a measure passes. Denote a simple weighted-voting game by $[q; w_1, w_2, \ldots, w_n]$.

Typically, simple weighted-voting games are restricted by the following properties: w_i is a nonnegative integer for every i and $q > \dfrac{\sum_{i=1}^{n} w_i}{2}$. When the weights are restricted to nonnegative integer values, then we will call these *discrete simple weighted-voting games.* Consider the motivational examples of a jury, committee, and shareholders of a company. Every member of a jury must vote "guilty" for a defendant in a criminal trial to be found guilty. If any of the 12 members of the jury vote "not guilty," then the jury is a hung jury and the defendant is found innocent. (Although if the jurors do not all agree on "guilty" or "not guilty," then the defendant of a trial may be re-tried.) Since a juror's vote is indistinguishable from another juror's vote, this jury can be modeled by the following simple weighted-voting game: $[12; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Notice that all voters' votes have the same weight.

There is not a unique simple weighted-voting game to model the jury's voting process. Indeed, $[78; 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$ also represents the trial by jury because all of the voters must agree in the affirmative to return a guilty charge and the sum of the weights of the voters is the quota, $q = 66$. Although all of the jurors' votes are weighted differently, it is clear that they must vote unanimously to arrive at a guilty verdict, as before.

Consider a committee with a chairperson and three other members where in order for the committee to pass a measure, the chairperson and at least two of the other members must vote "yes." This can be represented by the simple weighted-voting game $[4; 2, 1, 1, 1]$, where the vote of the chairperson has weight 2. Notice that a measure can be passed only if the quota of 4 is reached. This only can be met by having the chairperson and two other members of the committee vote "yes" (since $2 + 1 + 1 \geq 4$) or by having the chairperson and all three other members vote "yes" (since $2+1+1+1 > 4$). The simple weighted-voting game $[7; 3, 2, 2, 2]$ also models the relationship between the committee members' votes. In both games, a measure can only be passed if v_1 votes "yes."

Suppose that a company has three stockholders and that each share of stock grants the owner of the stock one vote. Assume that the three stockholders own 55, 30, and 15 shares and that a measure is passed if a majority of the votes are in favor of the measure. This can be represented by the simple weighted-voting game: $[51; 55, 30, 15]$. Although the numbers in this simple weighted-voting game seem natural, realize that $[501; 550, 300, 150]$ also models the voting relationship between the shareholders. A measure passes if v_1 votes "yes" and fails to pass if v_1 votes "no."

Exercise 1. A Hiring Committee consists of a Personnel Director, a Team Manager, and three team members. The committee agrees to hire an applicant if, at the minimum, the director and all three team members or the director, team manager, and two team members agree to hire the applicant. Construct a simple weighted-voting game to model this situation.

Exercise 2. The quota of a simple weighted-voting game is required to be greater than half of the sum of the weights of all of the voters. To see why this is the case, consider the simple weighted-voting game $[50; 50, 30, 20]$ and the measure "Voter 1 is the supreme ruler of the world." Does this measure pass? What about the measure "Voter 1 is not the supreme ruler of the world?" Does this measure pass? Explain why it is necessary to restrict the value of the quota.

Exercise 3. Suppose that the stock of a company splits. That is, assume that every share of stock is now worth 2 shares of stock. To preserve the relationship between the shareholders, what must be done to the quota? Explain.

Simple weighted-voting games provide a mathematical means to model political interactions. However, the mathematical framework presents opportunities not only to model, but formally to ask, and to answer, questions inspired by the political setting. The most pertinent question is: "who has political power?" From the examples, it is clear that a voter is better off having his vote carry a larger weight. But how is this related to political power? Power indices use the simple weighted-voting game structure along with insight about how political processes work to quantify the political power of players in a simple weighted-voting game. Before introducing the necessary

Permutations

v_1 $\textcircled{$v_2$}$ v_3	$w_1 < q$	$w_1 + w_2 \geq q$	
v_1 $\textcircled{$v_3$}$ v_2	$w_1 < q$	$w_1 + w_3 \geq q$	
v_2 $\textcircled{$v_1$}$ v_3	$w_2 < q$	$w_2 + w_1 \geq q$	
v_2 v_3 $\textcircled{$v_1$}$	$w_2 < q$	$w_2 + w_3 < q$	$w_2 + w_3 + w_1 \geq q$
v_3 $\textcircled{$v_1$}$ v_2	$w_3 < q$	$w_3 + w_1 < q$	
v_3 v_2 $\textcircled{$v_1$}$	$w_3 < q$	$w_3 + w_2 < q$	$w_3 + w_2 + w_1 \geq q$

Figure 1: Computing the Shapley-Shubik power index for $[3; 2, 1, 1]$.

ideas to define the Shapley-Shubik power index, we define terminology that represents extreme cases of the relationships between the voters. In fact, these terms are represented by the examples above.

Definition 2. A voter in a simple weighted-voting game

• is a *dictator* if she can pass a measure by voting "Yes," even if all other voters vote "No,"

• has *veto power* if she can defeat a measure by voting against it, even when all other voters support the measure, and

• is a *dummy voter* if the outcome of an election *never* depends on her vote.

Notice that the chairperson in the committee example has veto power while the first voter in the stockholder example is a dictator. Further, the other voters in the stockholder example are dummy voters. The jury example demonstrates that more than one voter, indeed all voters, may have veto power. Power indices provide a way to determine which simple weighted-voting games model the same situations. For applications of modeling real world situations with simple weighted-voting games, see COMAP, 2000.

The Shapley-Shubik power index focuses on the order of "yes" votes and who casts the deciding, or pivotal, vote. The pivotal voter has the power for this sequence of votes. The Shapley-Shubik power index of a voter is the number of times that a voter is pivotal over all possible sequences, or permutations, of the order of voters.

Consider the simple weighted-voting game $[3; 2, 1, 1]$. For the permutation, $v_1\ v_2\ v_3$, the second voter is the pivotal voter since $w_1 < q$ and $w_1 + w_2 \geq q$. A convenient method to compute the Shapley-Shubik power index is to list all of the permutations of the voters and to circle the pivotal voter for each ordering. All permutations of the three voters are listed in Figure 1 and the pivotal voters of the game $[3; 2, 1, 1]$ are indicated. Merely counting the number of times that each voter is circled yields the Shapley-Shubik power index. For $[3; 2, 1, 1]$, the Shapley-Shubik power index is 4:1:1.

Exercise 4. The Three Stooges meet regularly to discuss career options. Since Moe is the most recognizable and, in some sense, most essential Stooge, he has veto power on all business decisions. However, Moe is *not* a dictator. For Moe to pass a measure, either Larry or Curly has to vote "yes" also. Represent this situation with a simple weighted-voting game.

Exercise 5. Define equivalent simple weighted-voting games to be games where the voters have the same Shapley-Shubik power index. Find two such 3-voter games.

Exercise 6. For the simple weighted-voting game, $[q; w_1, \ldots, w_n]$, explain why the following is true: If the Shapley-Shubik power index of voter i is greater than the Shapley-Shubik power index of voter j, then $w_i > w_j$.

Exercise 7. For the simple weighted-voting game, $[q; w_1, \ldots, w_n]$, explain why the following is false: If $w_i > w_j$, then the Shapley-Shubik power index of voter i is greater than the Shapley-Shubik power index of voter j.

Exercise 8. Prove that a voter is a dictator in an n-player simple weighted-voting game if and only if his Shapley-Shubik power index is $n!$.

Exercise 9. Prove that a voter is a dummy voter if and only if her Shapley-Shubik power index is zero.

3 Possible Shapley-Shubik Power Indices

The Shapley-Shubik power index can be described by a function that takes a simple weighted-voting game with n voters to an n-tuple with nonnegative integer terms that sum to $n!$. More rigorously, the Shapley-Shubik power index can be defined by the value function on coalitions, although this introduces ideas that are not germane to this development (see *e.g.*, Shapley and Shubik, 1954 for deriving the Shapley-Shubik power index from the Shapley value, commonly used in game theory.) From our examples, we know that the Shapley-Shubik power index also depends on the quota. Rather than describe the Shapley-Shubik power index by a function with domain of n-tuples of weights together with a quota, we define separate functions for the total weight (*i.e.*, the sum of all voters' weights) and the quota.

Definition 3. A Shapley-Shubik power index is an image point of a function from the set of simple weighted-voting games with total weight w and quota q *into* the set of nonnegative integer solutions of

$$s_1 + s_2 + \cdots + s_n = n!$$

More mathematically, the Shapley-Shubik power index is given by $S_{q,w}$ where $S_{q,w} : D \to R$ with domain

$$D = \{(w_1, w_2, \ldots, w_n) \mid w_i \geq 0 \text{ and } w_i \in \mathbb{Z}, \text{ for all } i, \text{ and } \sum_{i=1}^{n} w_i = w\}$$

and range

$$R = \{(s_1, s_2, \ldots, s_n) \mid s_i \geq 0 \text{ and } s_i \in \mathbb{Z}, \text{ for all } i, \text{ and } \sum_{i=1}^{n} s_i = n!\}.$$

The definition and functional notation of the Shapley-Shubik power index naturally leads to purely mathematical questions that will also have implications on applications. First and foremost, we may want to know if, for fixed w and q, is the Shapley-Shubik power index a one-to-one and/or onto function? Realize that for a 3-voter simple weighted-voting game that the Shapley-Shubik power index maps into the set of nonnegative integer solutions of $s_1 + s_2 + s_3 = 6$; there are $\binom{8}{2} = 28$ possible image points. Which of these possible image points are in the range (for some w and q)?

Exercise 10. Determine the 28 *possible* image points for the Shapley-Shubik power index for 3-voter, simple weighted-voting games. Use a combinatorial argument to explain why there are 28 possible image points.

Exercise 11. Use a combinatorial argument to count how many possible image points there are for the Shapley-Shubik power index function when there are n voters.

Exercise 12. Prove that if $m{:}n{:}p$ is a Shapley-Shubik power index for a simple weighted-voting game, then all permutations of m, n, and p are possible power indices, too.

Exercise 13. Explain why a voter is a dictator if she is ever pivotal in a permutation where she appears in the first position.

The following examples display the type of analysis necessary to determine the range of the Shapley-Shubik power index.

Example 1. 3:2:1 is not a possible Shapley-Shubik power index.

Consider the orderings of the voters in Figure 1. Since voter 1 is not a dictator, then he cannot be the pivotal voter in the sequences $v_1\ v_2\ v_3$ and $v_1\ v_3\ v_2$ (see Exercise 13). Hence, voter 1 must be pivotal in three of the four orderings:

$$v_2\ v_1\ v_3 \qquad v_2\ v_3\ v_1 \qquad v_3\ v_1\ v_2 \qquad v_3\ v_2\ v_1.$$

More specifically, voter 1 must be pivotal in $v_2\ v_3\ v_1$ or $v_3\ v_2\ v_1$. Voter 1 being pivotal in either sequence implies that $w_2 + w_3 < q$. So, voter 1 must be pivotal in both of these orderings. Therefore, voter 1 must be pivotal in one of $v_2\ v_1\ v_3$ and $v_3\ v_1\ v_2$. By Exercise 6, it follows that $w_1 > w_2 > w_3$ and $w_2 + w_1 > w_1 + w_3$. Voter 1 must be pivotal in the ordering $v_2\ v_1\ v_3$.

Voter 2's power index is two. Hence, voter 2 must be pivotal in two of the three orderings:

$$v_1\ v_2\ v_3 \qquad v_1\ v_3\ v_2 \qquad v_3\ v_1\ v_2.$$

One of these must be $v_1\ v_2\ v_3$. This follows since voter 1 is pivotal in $v_2\ v_1\ v_3$, indicating that $w_2 + w_1 \geq q$. However, if voter 2 is pivotal in either $v_1\ v_3\ v_2$ or $v_3\ v_1\ v_2$ then voter 2 must be pivotal in both. Hence, a contradiction on voter 2's power index being 2. So, 3:2:1 is not a possible Shapley-Shubik power index.

Example 2. 3:3:0 is a possible Shapley-Shubik power index.

Since voter 3 is a dummy voter, he can never influence the outcome of an election. Since neither voter 1 nor voter 2 is a dictator, both voters must agree in the affirmative and vote "yes" for a measure to pass. In the sequence of "yes" votes, which ever of voter 1 or voter 2 appears later in the sequence is the pivotal voter. Both voter 1 and voter 2 are equally likely to appear later in the six orderings. Hence, both are pivotal voters in three of the six orderings. The simple weighted-voting game $[4; 2, 2, 1]$ has Shapley-Shubik power index 3:3:0.

Exercise 14. Realize that 2:2:2 is a valid Shapley-Shubik power index for certain values of w and q. However, the power index can be achieved by different relationships between the voters; one such way is unaminity rule while the other is majority rule. Explain.

Exercise 15. Explain why 5:1:0 is not a valid Shapley-Shubik power index.

Exercise 16. Can you ever get 2:2:2 for a Shapley-Shubik power index when $q = \frac{2}{3}w$?

Exercise 17. Determine which of the 28 points are images of the Shapley-Shubik power index for some 3-voter, simple weighted-voting game.

The analysis used in the previous two examples can be extended to any number of voters, albeit with some difficulty. However, thinking of which nonnegative integer solutions of $s_1 + s_2 + \cdots + s_n = n!$ are valid Shapley-Shubik power indices for any n is beneficial to solving the problem for $n = 4$. The following theorem only uses properties of the addition of nonnegative integers and permutations.

Theorem 1. *If α_k is the pivotal voter of the permutation $\alpha_1\alpha_2\ldots\alpha_n$ of voters, then voter α_k's power index is at least $(k-1)!(n-k)!$.*

Proof. Let $w(\alpha_k)$ be the weight of voter α_k. Assume that k satisfies $1 < k < n$. The permutation can be written as $\alpha_1\alpha_2\ldots\alpha_{k-1}\alpha_k\alpha_{k+1}\ldots\alpha_n$. Because α_k is the pivotal voter, $\sum_{i=1}^{k-1} w(\alpha_i) < q$ and $\sum_{i=1}^{k} w(\alpha_i) \geq q$. It follows that α_k is the pivotal voter for the permutation of voters $\beta_1\beta_2\ldots\beta_{k-1}\alpha_k\gamma_{k+1}\ldots\gamma_n$, where $\beta = \beta_1\ldots\beta_{k-1}$ is a permutation of $\{\alpha_1,\ldots,\alpha_{k-1}\}$ and $\gamma = \gamma_{k+1}\ldots\gamma_n$ is a permutation of $\{\alpha_{k+1},\ldots,\alpha_n\}$. There are $(k-1)!$ permutations β and $(n-k)!$ permutations γ. Hence, α_k is the pivotal voter in at least $(k-1)!(n-k)!$ permutations.

If $k = 1$, then α_1 is the pivotal voter in $\alpha_1\alpha_2\ldots\alpha_n$. It follows that $w(\alpha_1) \geq q$ and that α_1 is a dictator. This implies that α_1 is the pivotal voter for all $n!$ permutations of the n voters. And, for $k = 1$, $n! > (k-1)!(n-k)!$ is in agreement with the theorem.

For $k = n$, the voter α_n is pivotal only when he appears at the end of the sequence of voters, or equivalently, at the end of the permutation. This occurs $(n-1)!$ times which is equal to $(k-1)!(n-k)!$ for $k = n$. □

The theorem can be applied to eliminate certain potential image points of the Shapley-Shubik power indices.

Corollary 2. *For n voters, a non-dummy voter's Shapley-Shubik power index is at least $(\lceil \frac{n}{2} \rceil - 1)!(n - \lceil \frac{n}{2} \rceil)!$ where $\lceil \cdot \rceil$ represents the ceiling function.*

Corollary 3. *For $n \geq 4$ voters, no voter has an odd power index.*

Proof. By Theorem 1, a non-dummy, non-dictator voter's power index is the sum of terms $(k-1)!(n-k)!$. For $n \geq 4$, $(k-1)!(n-k)!$ is even. And, the sum of even numbers is even. And, dictators and dummies always have even power indices. □

Exercise 18. Describe which possible Shapley-Shubik image points are eliminated by the corollaries for $n = 4$?

4 Discrete Approach to Probabilistic Questions

Re-examine Figure 1 where the Shapley-Shubik power index is determined for $[3; 2, 1, 1]$. The inequalities present in the figure determine which voter is pivotal for a particular sequence of voters.

The set of all inequalities defines the Shapley-Shubik power index. Indeed, any simple weighted-voting game $[q; w_1, w_2, w_3]$ with $w_1 + w_2 + w_3 = w$ satisfying

$$w_1 < q \quad w_2 < q \quad w_3 < q$$

$$w_1 + w_2 \geq q \quad w_1 + w_3 \geq q \quad w_2 + w_3 < q$$

$$w_1 + w_2 + w_3 \geq q$$

will have the *same* Shapley-Shubik power index, $4 : 1 : 1$, as $[3; 2, 1, 1]$.

Definition 4. For fixed w and q, two n-voter simple weighted-voting games are *ss*-equivalent if they have the same Shapley-Shubik power index.

Exercise 19. Show that the set of all *ss*-equivalent, n-voter simple weighted-voting games form an equivalence class.

It becomes a counting problem to determine how many simple weighted-voting games with fixed w and q are in any equivalence class, as demonstrated by the following two examples.

Example 3. Determine the number of simple weighted-voting games of the form $[14; w_1, w_2, w_3]$ with $w = 20$ in the equivalence class of $6 : 0 : 0$.

Since voter 1 is a dictator, it follows that $w_1 \geq 14$. The number of simple weighted-voting games $[14; w_1, w_2, w_3]$ with $w = 20$ in the equivalence class of $6 : 0 : 0$ is the number of nonnegative integer solutions of $w_1 + w_2 + w_3 = 20$ where $w_1 \geq 14$. Equivalently, it is $\sum_{w_1=14}^{20} |S_{w_1}|$ where S_{w_1} is the set of nonnegative integer solutions to $w_2 + w_3 = 20 - w_1$. Using a simple counting argument, it follows that $|S_{w_1}| = \binom{20-w_1+1}{1} = 20 - w_1 + 1 = 21 - w_1$. Hence, the number of simple weighted-voting games $[14; w_1, w_2, w_3]$ with $w = 20$ in the equivalence class of $6 : 0 : 0$ is

$$\sum_{w_1=14}^{20} |S_{w_1}| = \sum_{w_1=14}^{20} 21 - w_1 = 7 \cdot 21 - \sum_{w_1=1}^{20} w_1 + \sum_{w_1=1}^{13} w_1 = 147 - 210 + 91 = 28.$$

The next example demonstrates the counting problem for the set of inequalities from Figure 1.

Example 4. Determine the number of simple weighted-voting games of the form $[14; w_1, w_2, w_3]$ with $w = 20$ in the equivalence class of $4 : 1 : 1$.

Determining the number of simple weighted-voting games in the equivalence class is equivalent to determining the number of nonnegative integer solutions that satisfy $w_1 + w_2 + w_3 = 20$ and the following inequalities

$$w_1 < 14 \quad w_2 < 14 \quad w_3 < 14$$

$$w_1 + w_2 \geq 14 \quad w_1 + w_3 \geq 14 \quad w_2 + w_3 < 14.$$

Since $w_1 + w_2 + w_3 = 20$, the inequalities $w_1 + w_2 \geq 14$, $w_1 + w_3 \geq 14$, and $w_2 + w_3 < 14$ can be rewritten as $6 \geq w_3$, $6 \geq w_2$, and $6 < w_1$, respectively. But, $6 \geq w_3$ and $6 \geq w_2$ imply that $w_1 \geq 8$. Hence, we are counting the number of nonnegative integer solutions of $w_1 + w_2 + w_3 = 20$ such that $8 \leq w_1 \leq 13$, $w_2 \leq 6$, and $w_3 \leq 6$.

Let $w_1^* = w_1 - 8$, $w_2^* = w_2$, and $w_3^* = w_3$. The system transforms to $w_1^* + w_2^* + w_3^* = 12$ such that $0 \leq w_1^* \leq 5$, $0 \leq w_2^* \leq 6$, and $0 \leq w_3^* \leq 6$. Determining the number of integer solutions to the equality satisfying the inequality constraints can be achieved through a use of the

Inclusion-Exclusion Principle (*e.g.,* see Brualdi, 1999). Let S be the set of nonnegative integer solutions of $w_1^* + w_2^* + w_3^* = 12$. Further, let P_1 be the property that $w_1^* > 5$ and let P_i be the property that $w_i^* > 6$, for $i = 2$ and 3. Define the set $A_i = \{(w_1^*, w_2^*, w_3^*) : (w_1^*, w_2^*, w_3^*) \in S$ and (w_1^*, w_2^*, w_3^*) has property $P_i\}$. Then, the number of nonnegative integer solutions of $w_1^* + w_2^* + w_3^* = 12$ such that $w_1^* \leq 5$, $w_2^* \leq 6$, and $w_3^* \leq 6$ is, by the Inclusion-Exclusion Principle,

$$|A_1' \cap A_2' \cap A_3'| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

where A' is the complement of set A. This simplifies to $\binom{14}{2} - \binom{8}{2} - \binom{7}{2} - \binom{7}{2} = 21$. There are 21 simple weighted-voting games with $w = 20$ and $q = 14$ *ss*-equivalent to $4 : 1 : 1$.

Assume that there is a uniform distribution over all simple weighted-voting games with w and q fixed. That is, all simple weighted-voting games $[q; w_1, w_2, \ldots, w_n]$ such that $w_1 + w_2 + \cdots + w_n = w$ are equally likely to occur. Then the size of the equivalence class of the simple weighted-voting games for a Shapley-Shubik power index naturally corresponds to the likelihood of picking a simple weighted-voting game at random and this game having that particular Shapley-Shubik power index. Consider the following extension of the previous example.

Example 5. Determine the likelihood of a simple weighted-voting game having Shapley-Shubik power index $4 : 1 : 1$ for $w = 20$ and $q = 14$.

There are $\binom{22}{2} = 231$ simple weighted-voting games $[14; w_1, w_2, w_3]$ with $w_1 + w_2 + w_3 = w = 20$ that have Shapley-Shubik power index of $4 : 1 : 1$; this is the same number of nonnegative integer solutions of $w_1 + w_2 + w_3 = w = 20$. If all of these games are equally likely to occur, then the probability of selecting any one at random is $\frac{1}{231}$. Hence, the probability of a simple weighted-voting game with $w = 20$ and $q = 14$ having Shapley-Shubik power index is $\frac{21}{231} = \frac{1}{11}$, since there are 21 simple weighted-voting games with $w = 20$ and $q = 14$ with Shapley-Shubik power index $4 : 1 : 1$.

The idea of considering the probabilities in which certain Shapley-Shubik power indices occur is a natural extension of the counting problem. Consider the possibility that there are a fixed number of shares of stock of a company, but that shareholders may sell their stock to other shareholders. We assume that all arrangements are equally possible. There are examples and exercises in COMAP (2000) that consider how many shares could be sold without changing the power index of the game.

Exercise 20. Determine how many 3-voter, simple weighted-voting games with $w = 20$ and $q = 14$ are *ss*-equivalent to $3 : 3 : 0$.

Exercise 21. Find a representative simple weighted-voting game with 4 voters that has Shapley-Shubik power index $6 : 6 : 6 : 6$ for $w = 20$ and $q = 11$. Compare this to a representative 4-voter game that has Shapley-Shubik power index $6 : 6 : 6 : 6$ for $w = 20$ and $q = 18$. Discuss the relationships between the pivotal voters in the two games.

5 Geometrical Interpretation of the Shapley-Shubik power index

A simple weighted-voting game $[q; w_1, w_2, \ldots, w_n]$ can be normalized by dividing the quota and players' weights by $w = w_1 + w_2 + \cdots + w_n$. The players' weights of the normalized game

$$\left[\frac{q}{w}; \frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right]$$

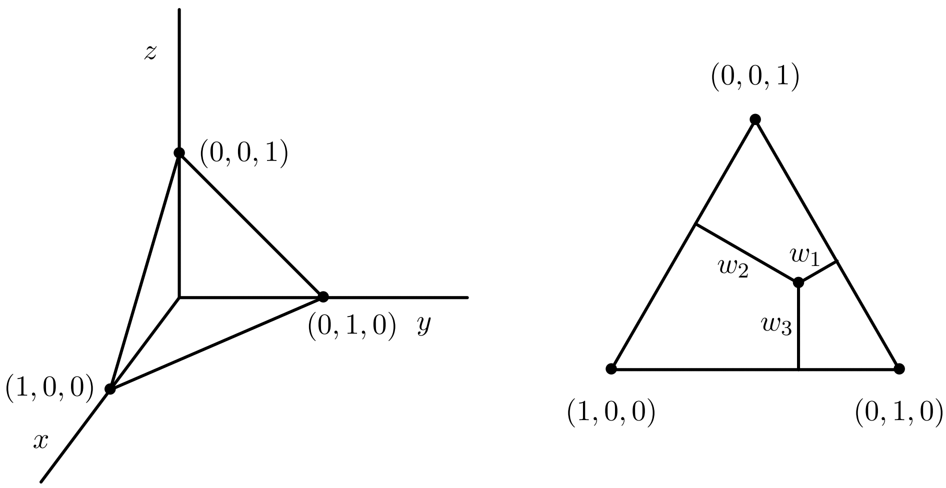

Figure 2: The plane $x+y+z = 1$ and the 2-simplex $\{(w_1, w_2, w_3) \mid w_1+w_2+w_3 = 1, w_1 \geq 0, w_2 \geq 0,$ and $w_3 \geq 0.\}$

can be considered geometrically as a point on the $(n-1)$-simplex, denoted S_{n-1}; the $(n-1)$-simplex is the set of nonnegative solutions to

$$x_1 + x_2 + \cdots + x_n = 1.$$

Since any $n-1$ values of x_i's define a point on the simplex, the simplex has dimension $n-1$. For a game with 3 players, the normalized weights of the three players can be viewed as a point on the 2-simplex. The 2-simplex is the intersection of the plane $x_1 + x_2 + x_3 = 1$ and the positive octant where $x_i \geq 0$ for all i; this forms a triangle as shown in Figure 2.

In Section 2, we restricted our attention to discrete simple weighted-voting games where weights were nonnegative integer values. In this section, we allow weights to be nonnegative real numbers; we call these *continuous simple weighted-voting games.* For the quota being a fixed percentage of the sum weight of all voters, a point on S_{n-1} represents an equivalence class of simple weighted-voting games for n players. This follows because many simple weighted-voting games have the same normalized representation. The simple weighted-voting game $[q; w_1, w_2, \ldots, w_n]$ is normalized to $\left[\frac{q}{w}; \frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right]$. The weights of the pre-normalized game can be viewed as the point $(w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$. The normalized weights are represented by the point $\left(\frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right)$ on the $(n-1)$-simplex. Geometrically, these two points are related since the line connecting the origin and $(w_1, w_2, \ldots, w_n)$ intersects the $(n-1)$-simplex at $\left(\frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right)$. Hence, the equivalence class of simple weighted-voting games represented by $\left(\frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right)$ is every point on the ray from the origin through $\left(\frac{w_1}{w}, \frac{w_2}{w}, \ldots, \frac{w_n}{w}\right)$.

The Shapley-Shubik power index of a simple weighted-voting game of n voters, $s_1 : s_2 : \cdots : s_n$, can be normalized, also. Specifically, the normalized power index $\frac{s_1}{n!} : \frac{s_2}{n!} : \cdots : \frac{s_n}{n!}$ is a point on the $(n-1)$-simplex since

$$\frac{s_1}{n!} + \frac{s_2}{n!} + \cdots + \frac{s_n}{n!} = 1$$

where $\frac{s_j}{n!} \geq 0$ for all j. The 10 possible Shapley-Shubik power indices for 3-voter simple weighted-voting games are 6:0:0, 0:6:0, 0:0:6, 4:1:1, 1:4:1, 1:1:4, 3:3:0, 3:0:3, 0:3:3, and 2:2:2 (this answers Exercise 3.8). Normalized, these power indices are graphed on the 2-simplex in Figure 3; they are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, $(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$, $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(0, \frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

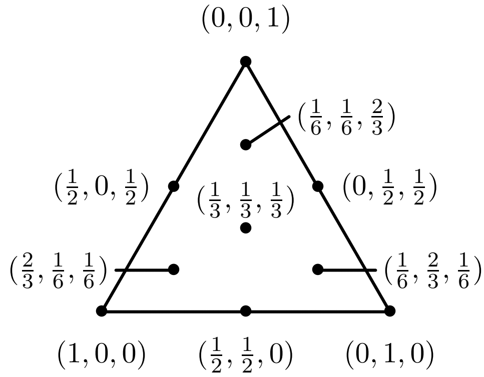

Figure 3: Possible Shapley-Shubik power indices on the 2-simplex.

Assume for the remainder of the paper that all weights, quotas, and Shapley-Shubik power indices are normalized, unless otherwise indicated. Realize that the normalized quota is in $\left(\frac{1}{2}, 1\right]$. The Shapley-Shubik power index can be viewed as a map from S_{n-1} to S_{n-1}, mapping a simple weighted-voting game (with a specified normalized quota q) to a Shapley-Shubik power index; denote this map by P_q. As discussed previously (*e.g.*, in Figure 1), the Shapley-Shubik power index of a given game $[q; w_1, w_2, \ldots, w_n]$ (assumed to be normalized) is determined by which sums of weights are greater than or equal to the quota. The collection of inequalities partition S_{n-1} into different regions where each point in a region has the same Shapley-Shubik power index.

Each point in a specific partition has the same Shapley-Shubik power index; thus, every point in a region has the same image under P_q. Graphing the boundaries of the regions, *i.e.*, all combinations of sums of weights equaling the quota, yields a geometric representation of the partitioning of the simplex by the Shapley-Shubik power index map.

For $n = 3$, the partitioning equations of S_2 are

$$w_1 = q, \;\; w_2 = q, \;\; w_3 = q, \;\; w_1 = 1 - q, \;\; w_2 = 1 - q, \text{ and } \;\; w_3 = 1 - q.$$

Due to the symmetry, there are 4 regions up to permutation on the set of players; these regions are described below. Summative data appears in Table 1 and the regions are pictured in Figure 4. Note that the shape of the regions is dependent on the quota. For the below descriptions, assume that the normalized quota is fixed.

Exercise 22. From Figure 1, it appears that $w_1 + w_2 = q$ should be one of the equations that partition the 2-simplex. Explain why $w_1 + w_2 = q$ is accounted for in the previous paragraph.

Case 1. (Dictator Regions R_1, R_2, and R_3). A point (w_1, w_2, w_3) is in region R_i if and only if voter i is a dictator, i.e. $w_i \geq q$. All of the points in R_i are mapped to the Shapley-Shubik power index with a one in position i and zeros elsewhere.

Case 2. (Regions R_4, R_5, and R_6). A point (w_1, w_2, w_3) is in region R_{i+3} if and only if voter i is a dummy voter and the other two voters have equal power. This occurs when $w_i + w_j < q$, $w_i + w_k < q$, and $w_j + w_k \geq q$ where i, j, and k are distinct voters in $\{1, 2, 3\}$. All of the points in region R_{i+3} are mapped to the Shapley-Shubik power index with a zero in position i and $\frac{1}{2}$ in the other two positions.

Region	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
SSPI	1:0:0	0:1:0	0:0:1	$0{:}\frac{1}{2}{:}\frac{1}{2}$	$\frac{1}{2}{:}0{:}\frac{1}{2}$	$\frac{1}{2}{:}\frac{1}{2}{:}0$	$\frac{2}{3}{:}\frac{1}{6}{:}\frac{1}{6}$	$\frac{1}{6}{:}\frac{2}{3}{:}\frac{1}{6}$	$\frac{1}{6}{:}\frac{1}{6}{:}\frac{2}{3}$	$\frac{1}{3}{:}\frac{1}{3}{:}\frac{1}{3}$

Table 1: Regions and their corresponding Shapley-Shubik power indices.

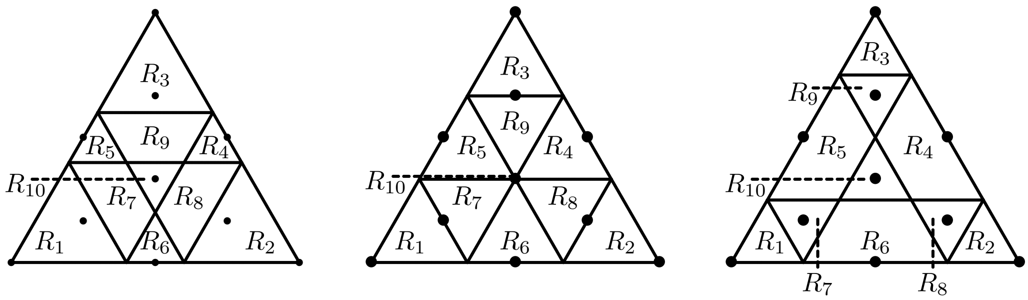

Figure 4: Shape of regions for $q < \frac{2}{3}$ (left), $q = \frac{2}{3}$ (middle), and $q > \frac{2}{3}$ (right).

Case 3. (Regions R_7, R_8, and R_9). A point (w_1, w_2, w_3) is in region R_{i+6} if and only if $w_i < q$, $w_i + w_j \geq q$, $w_i + w_k \geq q$, and $w_j + w_l < q$. All of the points in region R_{i+6} are mapped to the Shapley-Shubik power index with a $\frac{2}{3}$ in position i and $\frac{1}{6}$ in the other two positions.

Case 4. (Region R_{10}). A point (w_1, w_2, w_3) is in region R_{10} if all voters have equal power. This can occur in different ways depending on the quota. For $q \leq \frac{2}{3}$, a point is in this region if $w_i \leq 1-q$ for all voters i. Equivalently, $w_j + w_k \geq q$ for all voters j and k. For $q > \frac{2}{3}$, a point is in this region if $w_i + w_j < q$ for all voters i and j. Hence, a measure passes only if all three voters agree in the affirmative. All of the points in region R_{10} are mapped to the Shapley-Shubik power index $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

The above lengthy reasoning amounts to saying that any game lying in region R_i has the Shapley-Shubik power index as listed in Table 1.

Exercise 23. Normalize the simple weighted-voting game $[21; 7, 12, 13]$. Which region contains the normalized game? Determine the Shapley-Shubik power index of the game.

Exercise 24. Normalize the simple weighted-voting game $[3; 2, 1, 1]$. Which region contains the normalized game? Determine the Shapley-Shubik power index of the game.

Exercise 25. For $q = \frac{3}{4}$, find the Shapley-Shubik power index of the game corresponding to the point given in Figure 5.

Example 6. The normalized game from Example 4 is $[\frac{14}{20}; \frac{w_1}{20}, \frac{w_2}{20}, \frac{w_3}{20}]$. As in Example 4, consider the possible values of w_1, w_2, and w_3 so that the simple weighted-voting game has a Shapley-Shubik power index of $\frac{2}{3}{:}\frac{1}{6}{:}\frac{1}{6}$. There are a finite number of games with $w = 20$ (before normalization).

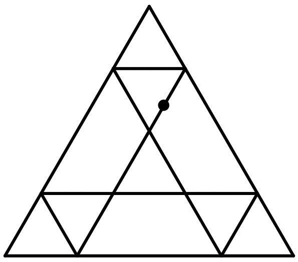

Figure 5: What is the Shapley-Shubik power index of the game represented above?

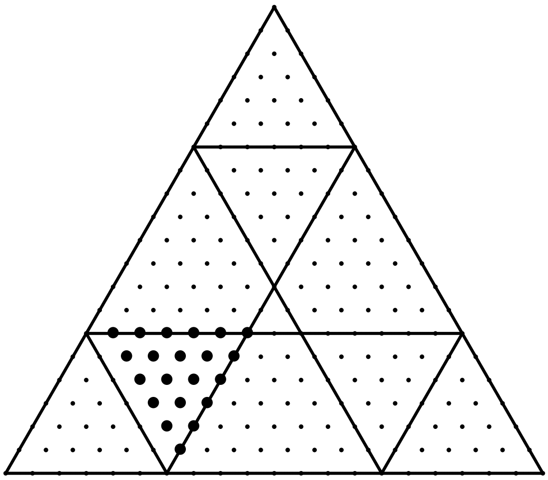

Figure 6: Lattice points of games that cover the 2-simplex.

Normalized, these games form a set of lattice points on the 2-simplex. The games that have the same Shapley-Shubik power index will lie in the same partition of the simplex.

The Shapley-Shubik power index of $\frac{2}{3}:\frac{1}{6}:\frac{1}{6}$ corresponds to region R_7 in Figure 4. Thus, Example 4 can be viewed as counting the lattice points that lie within region R_7. All 231 integer solutions to $\frac{w_1}{20} + \frac{w_2}{20} + \frac{w_3}{20} = 1$ are evenly spaced out in the simplex and are pictured in Figure 6. The 21 lattice points in region R_7 are darkened in Figure 6; this agrees with the 21 lattice points from the solution to Example 4.

In Example 4, if w is increased, then the total number of discrete games increases since the number of solutions to $w_1+w_2+w_3 = w$ increases. Indeed, there are $\binom{w+2}{2} = \frac{(w+2)(w+1)}{2}$ nonnegative integer solutions to $w_1 + w_2 + w_3 = w$. Consequently, the number of lattice points on the 2-simplex increases. As w increases without bound, the lattice points fill up the simplex. In particular, the lattice points that have the Shapley-Shubik power index of $\frac{2}{3} : \frac{1}{6} : \frac{1}{6}$ fill up region R_7. Although the number of games in region R_7 increases without bound as $w \to \infty$, the ratio of games in region R_7 to the total number of games approaches the ratio of the area of region R_7 to the area of the 2-simplex.

Region	R_1, R_2, R_3	R_4, R_5, R_6	R_7, R_8, R_9	R_{10}
Probability ($q < \frac{2}{3}$)	$(1-q)^2$	$(2q-1)^2$	$-8q^2+10q-3$	$(2-3q)^2$
Probability ($q = \frac{2}{3}$)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
Probability ($q > \frac{2}{3}$)	$(1-q)^2$	$-5q^2+8q-3$	$(1-q)^2$	$(3q-2)^2$

Table 2: Probabilities of regions for all values of the quota.

Assume that there exists a uniform distribution over the n-simplex. So, every point is equally likely to be selected as the weights of a simple weighted-voting game. For a fixed quota q, the likelihood that a particular Shapley-Shubik power index $\mathbf{s}$ occurs is merely the volume of the region that maps to $\mathbf{s}$ divided by the volume of the simplex. Although it is difficult to picture the higher dimensional simplices, even for $n = 4$, it is still possible to determine the likelihood of certain outcomes. For $n = 3$, it is quite easy to compute the probability of a simple weighted-voting game having a particular Shapley-Shubik power index. These probabilities are computed as functions of the quota in Table 2. We consider this limit process explicitly in the next example.

Exercise 26. Assume the normalized quota is $q < \frac{2}{3}$. Use basic geometry to verify the probability that a simple weighted-voting game selected at random from a uniform distribution over the 2-simplex has $0 : \frac{1}{2} : \frac{1}{2}$ as its Shapley-Shubik power index is $(2q-1)^2$. (*Hint*: Remember to divide the area of R_4 by the area of the 2-simplex.)

Exercise 27. For each of $q < \frac{2}{3}$, $q = \frac{2}{3}$, and $q > \frac{2}{3}$, verify that the sum of the probabilities that each region occurs is 1.

Example 7. Assume that the normalized quota q is greater than $\frac{2}{3}$ and that w is an integer. As w approaches infinity, we can determine the likelihood that a simple weighted-voting game $\left[q; \frac{w_1}{w}, \frac{w_2}{w}, \frac{w_3}{w}\right]$ selected at random has Shapley-Shubik power index $\frac{2}{3} : \frac{1}{6} : \frac{1}{6}$. As w increases, we expect that this likelihood will converge to $(1-q)^2$, the value from Table 2 for region R_7.

The inequalities that define region R_7 (from Case 3) are

$$\frac{w_1}{w} < q, \quad \frac{w_1 + w_2}{w} \geq q, \quad \text{and} \quad \frac{w_1 + w_3}{w} \geq q.$$

Using a little algebra, these inequalities can be rewritten as

$$w_1 < qw, \quad w_3 \leq (1-q)w, \quad \text{and} \quad w_2 \leq (1-q)w.$$

Because $w_3 \leq (1-q)w$ and $w_2 \leq (1-q)w$, it follows that $w_2 + w_3 \leq 2(1-q)w$ or $w_1 + w_2 + w_3 \leq 2(1-q)w + w_1$. This simplifies to $(2q-1)w \leq w_1$. This bounds w_1 from below. The number of simple weighted-voting games in region R_7 for fixed w is the number of nonnegative integer solutions of

$$\begin{aligned} w_1 + w_2 + w_3 &= w \text{ subject to} \\ (2q-1)w \le w_1 &< qw \\ w_2 &\le (1-q)w \\ w_3 &\le (1-q)w. \end{aligned}$$

For different values of w, the products qw and $(1-q)w$ may or may not be integers. Since we are concerned with the limit process, we will assume that qw and $(1-q)w$ are always integer values. This assumption does not change the calculation. Indeed, one could look at the case where these values are integers or are not integers. In both cases, the limits converge to the same value.

As in Example 4, we can use the Inclusion-Exclusion Principle to count the number of nonnegative integer solutions to the above equality with inequality constraints. However, it is simpler to transform the system above by a change of variables, as would be done in the Inclusion-Exclusion Principle, but then consider the transformed system geometrically. Let $w_1^* = w_1 - (2q-1)w$, $w_2^* = w_2$, and $w_3^* = w_3$. The above system becomes

$$w_1^* + w_2^* + w_3^* = 2(1-q)w \text{ s.t. } w_1^* < (1-q)w \text{ and } w_i^* \le (1-q)w_i \text{ for } i = 2, 3.$$

Hence, the number of nonnegative integer solutions to this new system is equal to the number of solutions to the original system.

Notice that all of the inequalities of the new system involve $(1-q)w$, exactly half of $2(1-q)w$, the sum of the variables. This allows us to easily visualize the solutions, as in Figure 7. The darkened dots represent solutions; this region forms an equilateral triangle of lattice points with $(1-q)w$ points on every side. So, the number of solutions is the sum of the first $(1-q)w$ positive integers. Hence, there are $\binom{(1-q)w+1}{2}$ solutions.

To determine the ratio of solutions to possible simple-weighted voting games, we merely divide $\binom{(1-q)w+1}{2}$ by the number of simple-weighted voting games with total weight w. There are $\binom{w+2}{2}$ such games. Hence, the limit of the number of games that have Shapley-Shubik power index of $\frac{2}{3} : \frac{1}{6} : \frac{1}{6}$ is

$$\lim_{w\to\infty} \frac{[(1-q)w][(1-q)w+1]}{(w+2)(w+1)} = \lim_{w\to\infty} \frac{(1-q)^2w^2 + (1-q)w}{w^2+3w+2} = (1-q)^2.$$

This value agrees with the entry in Table 2 for region R_7 when $q > \frac{2}{3}$.

Exercise 28. Use Table 2 to determine the probability that a randomly selected, continuous simple weighted-voting game with normalized quota $q = 0.7$ will have a Shapley-Shubik power index of $0 : \frac{1}{2} : \frac{1}{2}$.

Exercise 29. For normalized quota $q = 0.7$, determine the number of discrete simple weighted-voting games with a Shapley-Shubik power index of $0 : \frac{1}{2} : \frac{1}{2}$ and $w = 20$. Now determine the probability that a randomly chosen discrete simple weighted-voting game with normalized quota 0.7 and $w = 20$ has a Shapley-Shubik power index of $0 : \frac{1}{2} : \frac{1}{2}$. If w were increased to 100, how would you expect the probability to change?

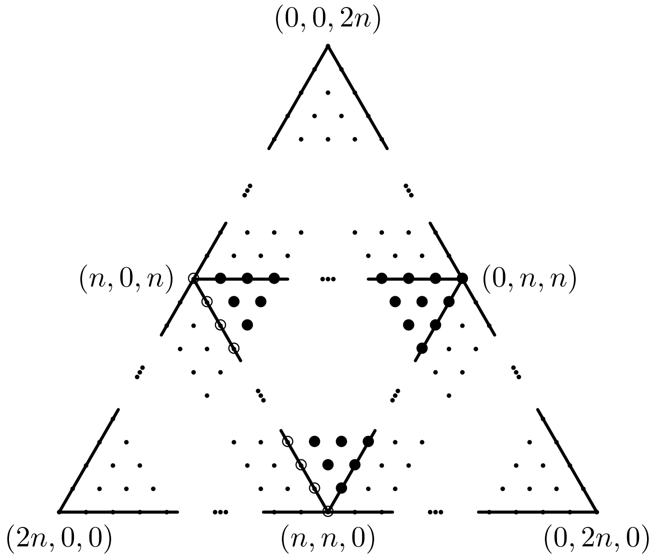

Figure 7: Transformed system from Example 7 where $n = (1 - q)w$.

We have linked the discrete and continuous simple weighted-voting games through limits. We have also come full circle. Our initial mathematics questions were motivated by the modeling of political interactions. We now have new tools to ask and answer questions about the likelihood of different outcomes. As an example, the next proposition applies the information from Table 2 and considers it in context of the modeling of political science. The corollary is a direct consequence of the proof of the proposition.

Proposition 4. *The probability of voter i being a dictator in an 3-voter simple weighted-voting game with normalized quota q is $(1-q)^2$.*

Proof. Let $i = 1$. Voter 1 is a dictator if and only if $w_1 \geq q$. The region on the simplex satisfying $w_1 \geq q$ is a triangle similar to the 2-simplex where the two triangles share the vertex $(1, 0, 0)$. The intersection of the hyperplane $w_1 = q$ and the simplex is the line segment with endpoints $(q, 1-q, 0)$ and $(q, 0, 1-q)$. The ratio of corresponding sides of the smaller simplex to the larger is $\sqrt{2}(1-q) : \sqrt{2}$ or $(1-q)$. By Euclid VI.19, the ratio of the areas of the smaller to the larger simplices is $(1-q)^2$. By symmetry, the proposition is true for $i = 1, 2$, or 3. □

Corollary 5. *The probability of there being a dictator in a 3-voter simple weighted-voting game with normalized quota q is $3(1-q)^2$.*

The proposition can be extended for any n using the relationship between the volume of similar n-dimensional regions. Haines and Jones (2002) contains the extension and applications of the techniques and perspectives of this paper to power indices and apportionment methods.

6 Guidelines for Use

This article naturally begins at an introductory level and offers more challenging aspects as it evolves. As a rule of thumb, earlier material has been used in multiple settings: a general education requirement course, an undergraduate applied combinatorics and graph theory course, and a graduate course in combinatorics, as well as in independent study courses. As the material becomes more difficult, it is accessible to fewer students. The following descriptions offer our suggestions on how to use this material for different courses, including the aforementioned, as well as a course in mathematical modeling. These guidelines are based on the mostly successful adaptations of this material in different level courses at Montclair State University from Spring 1999 to Spring 2002.

For a general education requirement course or mathematics for liberal arts course, the topics and treatment in the first two sections are appropriate. We suggest beginning with specific stories about political processes and asking how such processes might be modeled and who has the political power. After defining simple weighted-voting games, the students should be able to handle the subtleties of the constraint condition on the quota, as considered in Exercise 2. In such courses, the primary goal should be for the students to understand the range of applications of mathematics. For this reason, it is important to focus on modeling real and fictitious scenarios, as considered in the second section. Indeed, students should be asked to model different political institutions, such as the US Congress, European Union, *etc.* However, students can easily understand aspects of modeling, such as how changes in weights must be accompanied by a change in the quota, as in Exercise 3.

Students can become quite adept at translating the words and descriptions of a political process (specifically, what constitutes a coalition that can pass a measure) into mathematical conditions and ultimately simple weighted-voting games. One such problem is Exercise 4. Students should realize that there is more than one way to model the same situation. This is a good opportunity to discuss the idea of two simple weighted-voting games being equivalent, having the same sets of players being winning, losing, and blocking coalitions. Students also can give the conditions for a measure to pass, if given a simple weighted-voting game, although we do not provide any exercises of this form. Such examples can be found in Lampert 1988 and COMAP 2000. The definitions of a dictator, veto power, and a dummy voter are easy for students to understand. Translating these concepts into mathematics after defining the Shapley-Shubik power index is a little more difficult for students, but definitely do-able. For example, Exercises 8 and 9 can be approached by first considering the case where there are 3 voters. Students can usually get the results for $n = 3$ alone or in a group, but may need some assistance to make the jump for general n. More generally, students should be able to understand that a player whose vote has a larger weight will be at least as powerful (as defined by the Shapley-Shubik Power Index) as a player whose vote has a smaller weight. Of course, when introducing the Shapley-Shubik power index, there are nice opportunities to discuss and motivate permutations and factorial notation.

Although the more rigorous definition of the Shapley-Shubik power index as a function is inappropriate for most general education courses, some of the exercises and concepts in the third section are appropriate if pitched at the right level. For example, Exercises 10 and 11 may need some explanation, as well as examples of possible solutions. By having the students focus on the patterns of circles that can occur when determining the pivotal voter for all permutations of voters (as demonstrated in Figure 1), students can readily answer Exercises 12 and 13. After tallying the results from different examples where there are 3 voters, it is reasonable to focus on the possible indices that do not appear to represent the power of players in a simple weighted-voting game. The required analysis is a little more subtle, but can be accomplished by students in a mathematics for liberal arts course. For example, Exercise 15 should not be too difficult. It may be useful to explain why 3:2:1 is not a valid Shapley-Shubik power index (Example 1).

For more advanced courses, the material that is covered in a general education course is still applicable; it can be covered much more quickly and efficiently. If the advanced course is a course in mathematical modeling, more time should be spent on modeling different situations. The guidelines for an upper level applied combinatorics course are suitable for those interested in adapting this material to a class in modeling. However, there are more opportunities to challenge the students by having the students read material from works in mathematical political science that model different political institutions and discuss how the mathematics indicates certain properties of the institution's simple weighted-voting game. The end of the introduction cites appropriate material for modeling students to read, with some assistance.

For an undergraduate applied combinatorics course, after introducing simple weighted-voting games and the Shapley-Shubik power index, we suggest explicitly demonstrating the link between computing the power index and the equation with the inequality constraints (as is done at the beginning of Section 4). This naturally leads to determining the equivalence class of all games with a fixed weight that are ss-equivalent, as defined in Definition 4. The material at the end of Section 3 that generalizes the results for that section can be taught before or after the other material, or even omitted. However, it is natural to consider what consequences follow for simple weighted-voting games with more than 3 voters.

There are many variations on the theme of determining the number of simple weighted-voting games that are in the same equivalence class (and have the same Shapley-Shubik power index). However, in all such cases, it is necessary to emphasize setting up the system of inequality constraints and how manipulating the constraints assists in finding the minimal constraints. As demonstrated in the text, some of the constraints may end up being redundant (as in Example 4). Once the inequality constraints are determined, the problem becomes solvable by counting methods. Example 4 demonstrates how changes of variable simplify the calculations. Many of these problems will require the use of the Inclusion-Exclusion Principle. We suggest that the Inclusion-Exclusion Principle, as well as determining the number of nonnegative integer solutions to an equality with inequality constraints, be introduced before determining the number of simple weighted-voting games in the same ss-equivalence class. Alternatively, the basics of simple weighted-voting games and the Shapley-Shubik power index can be introduced earlier in the term and more advanced problems introduced as the skills are introduced. This approach has been successful in a masters level course in combinatorial mathematics at Montclair State University.

Although the counting problems can be taught without the geometry that follows in Section 5, we suggest that the geometry become the focus of instruction. The inequality constraints that determine if a simple weighted-voting game is in a particular equivalence class have obvious geometrical implications, although it may not be so obvious to mix the geometrical perspective in the discrete problem of counting the number of solutions to an equation with inequality constraints. Undergraduates and graduate students should be able to understand how the inequality constraints partition the simplex into different regions; indeed, the four cases (and permutations) should be seen to exhaust the simplex. Exercises 23-25 can be used to reinforce the concepts.

Determining the likelihood that a simple weighted-voting game with a fixed total weight is selected at random has a particular Shapley-Shubik power index is a minor extension of the previous ideas. However, it is a little more challenging to show that this likelihood approaches the ratio of the area of the partition region to the area of the simplex, as demonstrated in Example 6. Our solution to Example 6 simplifies the calculation by reconfiguring the problem geometrically. A more direct Inclusion-Exclusion Principle proof is also possible and may be more assessible and understandable for undergraduates. Students should have the intuition that as the total weight increases, then the number of "dots" in the simplex increases and fills up the region. For a modeling course, probabilistic questions can be related to a player being a dictator, as in Proposition 4 and Corollary 5. These can easily be extended for any number of voters.

We have used problems comparable to Example 6 as the basis for a presentation by a masters level student, as part of a graduate combinatorial mathematics course. Another student presented geometrical approaches to the 4-player simple weighted-voting game, where he viewed layers of the tetrahedron/3-simplex as triangles/2-simplices. By embedding the discrete problem into Euclidean space, the students may feel more comfortable and can develop their geometrical thinking.

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Biographical Sketches

Matthew J. Haines received a doctorate in arithmetic number theory from Lehigh University in 1994. Although he has no rigorous proof of his lack of knowledge in the area of voting theory or his total bewilderment of deep connections between the discrete and the continuous, he hopes to demonstrate these hypotheses by publishing as many articles in the area as possible.
Michael A. Jones received his doctorate in game theory from the Mathematics Department at Northwestern University in 1994. He is a firm believer that research and teaching can feed off of one another: research makes teaching more effective and interesting for students while teaching focuses ones attention on a subject which acts as a filter for research ideas. His research interests are in the mathematics of the social sciences, including political science, economics, and psychology. He is particularly interested in analyzing discrete problems with continuous mathematics, as this article suggests.