SIMPLE WEIGHTED-VOTING GAMES AND THE GEOMETRY BEHIND PARADOXES OF VOTING POWER

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Michael A. Jones
Department of Mathematical Sciences
Montclair State University
Upper Montclair, NJ 07043
jonesm@mail.montclair.edu

Voting, Simple Voting, and Simple Weighted-Voting Games

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- Simple Voting Game: $v(S) - v(S/i) = 0$ or 1
- Simple Weighted-Voting Game
  - Finite set of voters $N = \{1, 2, \ldots, n\}$
  - Voter i's vote carries weight w_i such that $w = \sum_{i=1}^{n} w_i$
  - There exists a quota q where $\frac{w}{2} < q \leq w$
  - For $S \subseteq N$,

$$v(S) = \begin{cases} 0 & \text{if } \sum_{i \in S} w_i < q, \\ 1 & \text{if } \sum_{i \in S} w_i \geq q. \end{cases}$$

Partitioning the Domain into Winning and Losing Coalitions

- $(w_1, w_2, \ldots, w_n)$ can be viewed as a point on the simplex where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = w$
- Winning and losing coalitions . . .

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- What is the effect of this equation?
  - Equations of the form $\sum_{i \in S} w_i = q$ partition the simplex

Example: Partitioning the Simplex for 3-Player Games

- (w_1, w_2, w_3) such that $w_i \geq 0$ and $w_1 + w_2 + w_3 = w$

$$w_1 = q \qquad w_2 = q \qquad w_3 = q$$

$$w_1 + w_2 = q \quad w_1 + w_3 = q \quad w_2 + w_3 = q$$

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$$w_3 = w - q \quad w_2 = w - q \quad w_1 = w - q$$

Hyperplanes Partition the Simplex into Regions: The Case of Three Voters

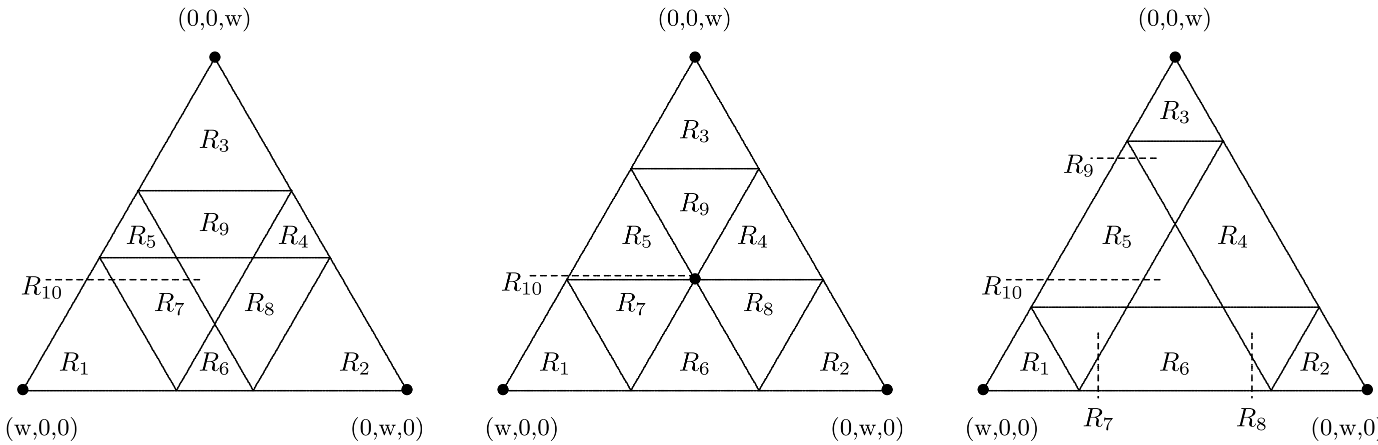

Figure 1: Shape of regions for $\frac{1}{2} < \frac{q}{w} < \frac{2}{3}$ (left), $\frac{q}{w} = \frac{2}{3}$ (middle), and $\frac{2}{3} < \frac{q}{w} < 1$ (right).

Region	R_i for $i = 1$ to 3	R_{i+3} for $i = 1$ to 3	R_{i+6} for $i = 1$ to 3	R_{10}
MWCs	$\{i\}$	$N/\{i\}$	$\{i, j\}, \{i, k\}$ where $i \neq j \neq k$	$\{1, 2\}, \{1, 3\}, \{2, 3\}^*$ or $\{1, 2, 3\}^{**}$

Table 1: Regions and their corresponding minimal winning coalitions (MWCs). The coalition structure for R_{10} depends on whether $q \leq \frac{2}{3}^*$ or $q > \frac{2}{3}^{**}$.

A Discrete Example using the Banzhaf Power Index

Find the power of the voters in the game [14; 12, 5, 3] using the Banzhaf Power Index:

$$B_{14}(12,5,3)_i = \sum_{S \subseteq N} [v(S) - v(S/i)]$$

- $\lambda_S = 1$ for the Banzhaf index [1]

- Other commonly used power index: Shapley-Shubik power index [19] where $\lambda_S = (\mid S \mid -1)!\,(n- \mid S \mid)!$

- What region is this game in?

A Discrete Example: Counting games in a region

from Haines and Jones [8]

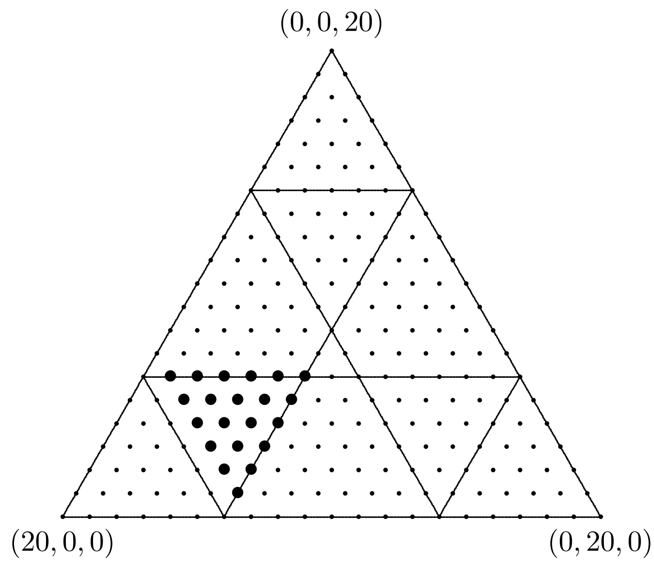

Figure 2: Lattice points of games that cover the 2-simplex.

More General Power Indices

- Power Indices must satisfy the following conditions:

  1. (Invariance under isomorphism) If there is an isomorphism of simple voting games that maps the winning coalitions of $\mathcal{W}$ to $\mathcal{W}'$ and voter a to voter a', then $(P_q(\mathcal{W}))_a = (P_q(\mathcal{W}'))_{a'}$.
  2. (Internal isomorphism) If two voters are members of the same winning coalitions, then they have the same power.
  3. (Dummy voter) If a voter a is never part of a minimal winning coalition, then voter a's power is 0.

Short History of the Literature on Power Indices

Applications

- Examining existent institutions: International Monetary Fund [5, 14], the Electoral College [15], the European Union Council of Ministers [9, 13], and the Israeli Knesset [12].

- Part of debate about the design of new institutions: new members in the EU [20, 21].

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Counterintuitive results

- *paradox of redistribution* [5, 17], the *donor and transfer paradoxes* [6], the *paradox of quarreling members* [11], the *paradox of a new member* [2, 3], and the *paradox of large size* [2, **?**]

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• Analyzing the domain and its geometry results in a classification of paradoxes of voting power (Jones [10])

– Geometry of 3 voters is sufficient to under the geometry behind the paradoxes

– Geometry that induces the paradoxes: passing a hyperplane, changing the partition, and changing dimensions

Passing Hyperplanes

Paradox of Redistribution

- Fischer and Schotter [7]; Schotter [17]: a power index exhibits this paradox if a voter's weight increases and its power goes down or if its weight decreases but its power goes up

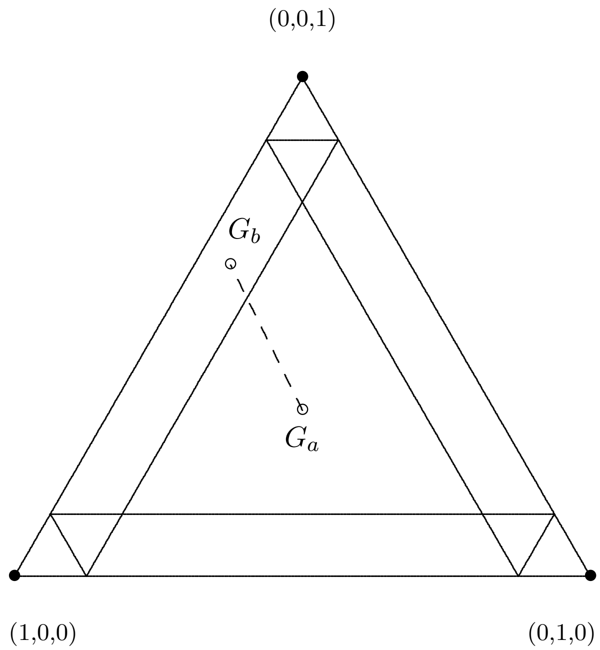
Donation Paradox

- Felsenthal and Machover [6]: "redistribute" between two voters

Fattening Paradox

- Felsenthal and Machover [6]: An increase in a voter's weight with other players' weights fixed results in a decrease in power
  - Example: $[8; 4, 4, 1, 1, 1] \rightarrow [8; 5, 4, 1, 1, 1]$

Paradox of Redistribution



$G_a = [\frac{7}{8}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ with power index $\frac{1}{3}$:$\frac{1}{3}$:$\frac{1}{3}$

$G_b = [\frac{14}{16}; \frac{5}{16}, \frac{1}{16}, \frac{10}{16}]$ with power index $\frac{1}{2}$:0:$\frac{1}{2}$

Figure 3: The Paradox of Redistribution as an effect of passing a hyperplane. When $G_a \to G_b$, player 1's weight decreases but its power increases. When $G_b \to G_a$, player 1's weight increases but its power decreases.

Partition Effects: Changing the Size, Shape, or Number of Parts of the Partition

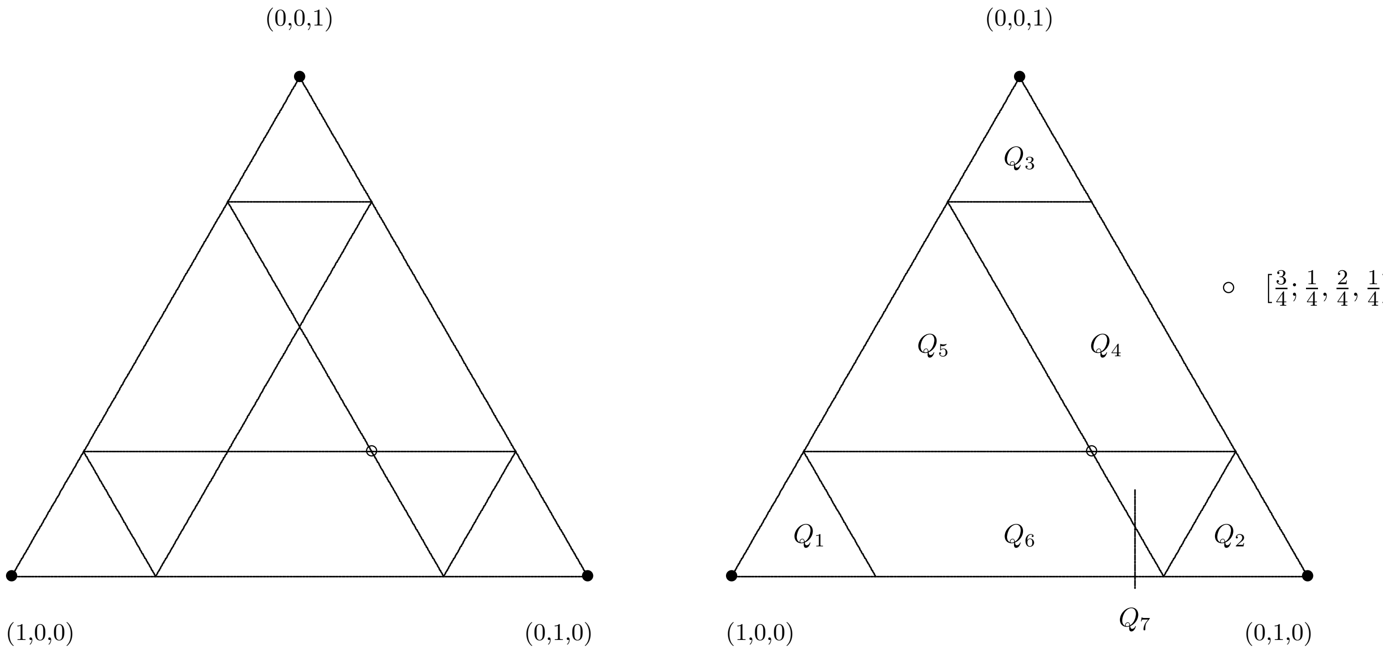
Paradox of Quarreling Members

- Kilgour [11]: quarreling members refuse to be in winning coalitions together; although options restricted, power increases

Quota Effect on "Largest" Voter's Power

- Voter with largest weight "should" benefit from a decrease in the quota

Paradox of Quarrelling Members



Power under Banzhaf P.I.: $\frac{1}{5}$:$\frac{3}{5}$:$\frac{1}{5}$

Power under Modified Banzhaf P.I.: $\frac{1}{4}$:$\frac{1}{2}$:$\frac{1}{4}$

Figure 4: The Paradox of Quarreling Members: Players 1 and 3 quarrel resulting in the removal of a hyperplane that decreases the number of parts in the partition.

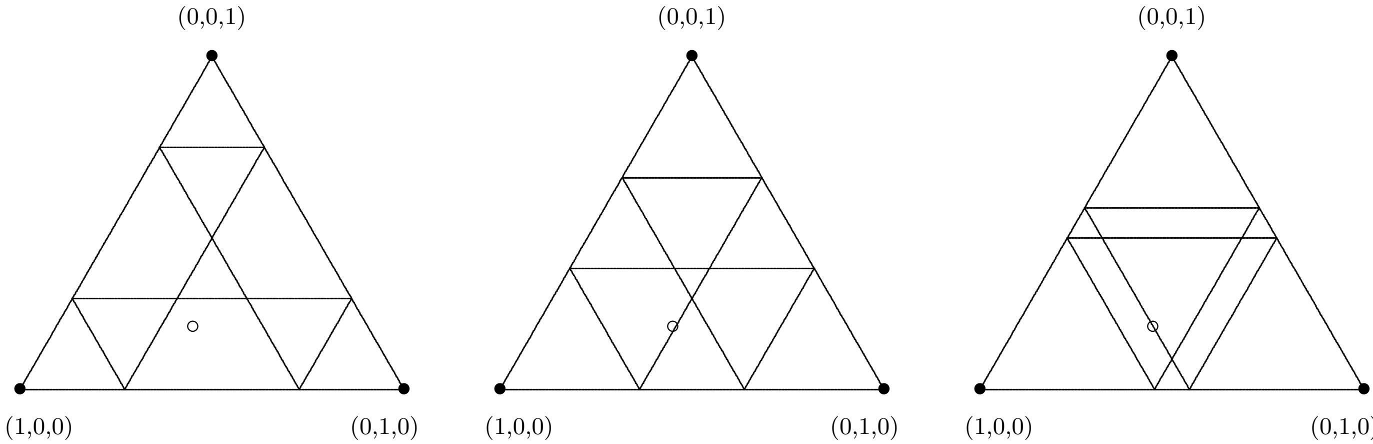


Figure 5: As the quota decreases from $q_1 = \frac{8}{11}$ (left), $q_2 = \frac{7}{11}$ (middle), and $q_3 = \frac{6}{11}$ (right) the weights $\frac{5}{11}, \frac{4}{11}, \frac{2}{11}$ form a game in region R_6, R_7, and R_{10}, respectively. Banzhaf P.I. from left to right: $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{1}{3}$.

Dimensional Effects: Increasing/Decreasing the Number of Voters

Paradox of a New Member

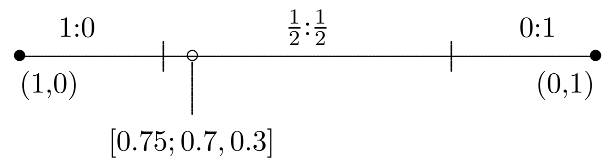
- Brams and Affuso [3], Brams [2]: an introduction of a new member can increase the power of one of the original members
- Felsenthal and Machover's [6] formulation: $[q; x_1, x_2, \ldots, x_n] \rightarrow [q; y_1, y_2, \ldots, y_n, y_{n+1}]$ where $y_{n+1} \in (0, 1]$ and $y_i = (1 - y_{n+1})x_i$ for $i \leq n$

Paradox of Large Size

- Shapley [**?**]; Brams [2]: one voter annexes another, yet its power decreases

- Saari and Sieberg [16]: Possible to flip rankings

Paradox of a New Member



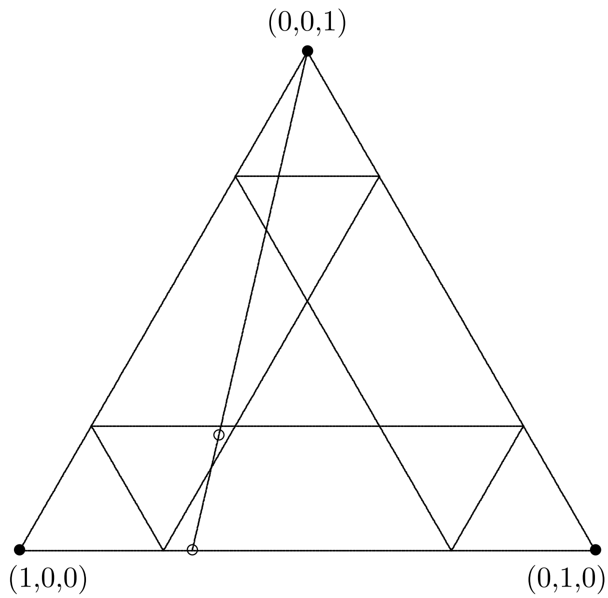


Figure 6: The Paradox of a New Member: Voter 1's power increases despite the introduction of a new voter.

Paradox of Large Size

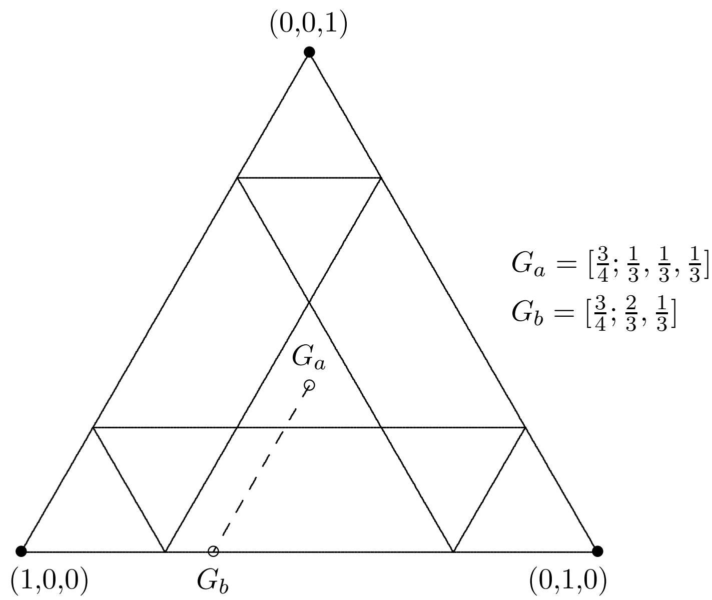


Figure 7: The Paradox of Large Size: Player 3 coalesces with Player 1 and their cumulative power decreases.

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