GEOMETRY OF PAIRWISE COMPARISONS:

Applications, Examples, and Techniques

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Pairwise Comparisons of Three Candidates

Assume there are 3 candidates, A, B, and C. For a profile of voters $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$, define

$$F_3 : Si(6) \to [-1, 1]^3 \text{ by } F_3(\mathbf{p}) = (d_{A \succ B}, d_{B \succ C}, d_{C \succ A})$$

where $d_{c_i \succ c_j}$ is the difference in the percentage of voters that prefer candidate c_i to c_j and the percentage of voters that prefer c_j to c_i.

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$$F_3(\mathbf{E}_1) = (1, 1, -1) \quad F_3(\mathbf{E}_2) = (1, -1, -1) \quad F_3(\mathbf{E}_3) = (1, -1, 1)$$
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Geometry of Pairwise Votes: Representation Cube

- Six of the eight vertices of the $[-1,1]^3$ form the corner points of a convex hull.
- The two missing vertices of the cube: $(1,1,1)$ and $(-1,-1,-1)$.

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- Visually: A truncated cube

Geometry of Pairwise Votes: Representation Cube

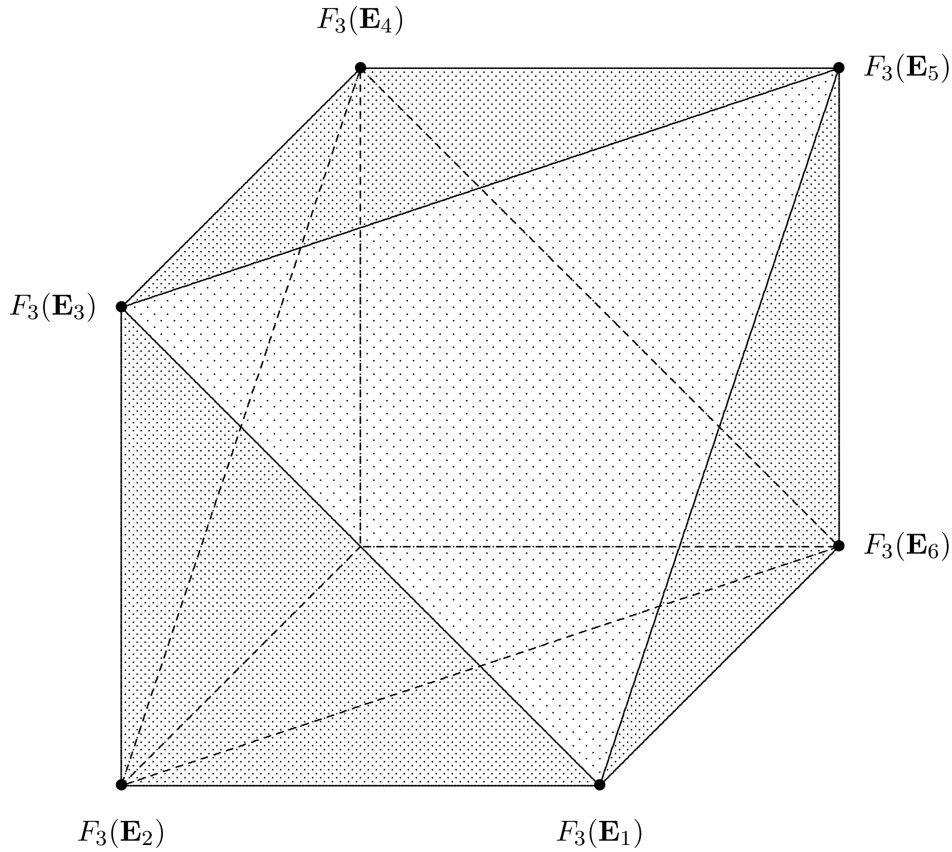

Figure 1: Representation cube for 3 candidates.

Geometry of Pairwise Votes: Representation Cube

- Two types of Condorcet cycles:

$$
\begin{aligned}
A \succ B \succ C \succ A \quad &\Rightarrow \quad (d_{A\succ B} > 0, d_{B\succ C} > 0, d_{C\succ A} > 0) : \text{positive cycle} \\
B \succ A \succ C \succ B \quad &\Rightarrow \quad (d_{A\succ B} < 0, d_{B\succ C} < 0, d_{C\succ A} < 0) : \text{negative cycle}
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• Let $x_1 = d_{A\succ B}, x_2 = d_{B\succ C},$ and $x_3 = d_{C\succ A}$. What is the extreme positive cycle? That is, the cycle that has the largest minimum margin of victory among the three pairwise elections? Equivalently, this point is in the positive orthant and satisfies

$$\max \min_{1\leq i\leq 3} x_i$$

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• Implication: There does not exist a cycle in which each candidate defeats another by receiving more than $\frac{2}{3}$'s of the vote.

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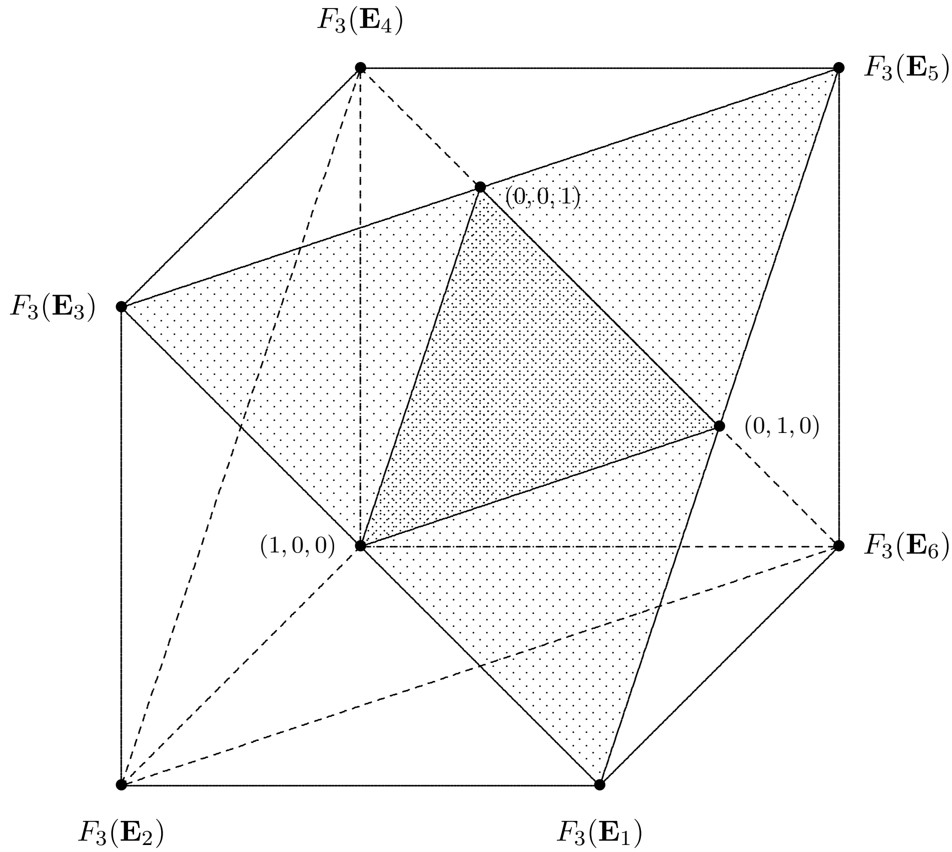

Figure 2: Intersection of positive orthant and triangular boundary of truncated cube (T_1).

Geometry of Pairwise Votes: Representation Cube

• The extreme positive cycle $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is on the triangular face of the truncated cube. What about points on this boundary?

– All points on the triangular boundary (T_1) are derived by a convex combination of voters of type 1, 3, and 5.

– The convex combination is unique. Hence, there is only one profile that yields the outcome.

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• Similarly, for extreme negative cycles . . .

– All points on the triangular boundary (T_2) are derived by a convex combination of voters of type 2, 4, and 6.

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– All points on the triangular boundary (T_2) are derived by a convex combination of voters of type 2, 4, and 6.

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• Interior points of the Representation Cube?

– Profiles satisfy 4 equations with 6 unknowns; 2 free variables. Hence, there exists a two-dimensional linear subspace of profiles in $Si(6)$ that support interior outcomes.

Geometry of Pairwise Votes: Representation Cube

Representing profiles in the Representation Cube

– Split the profile into odd and even types:

$$\mathbf{o} = (p_1, 0, p_3, 0, p_5, 0) \text{ and } \mathbf{e} = (0, p_2, 0, p_4, 0, p_6)$$

– Scale to be "valid" profiles: $\frac{1}{d}\mathbf{o}$ where $d = p_1 + p_3 + p_5$ and $\frac{1}{1-d}\mathbf{e}$

– $F_3(\frac{1}{d}\mathbf{o}) = \alpha \in T_1$ and $F_3(\frac{1}{1-d}\mathbf{e}) = \beta \in T_2$

– By convexity, $F_3(\mathbf{p}) = d\alpha + (1-d)\beta$. So, $F_3(\mathbf{p})$ is on the segment between α and β.

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– By convexity, $F_3(\mathbf{p}) = d\alpha + (1-d)\beta$. So, $F_3(\mathbf{p})$ is on the segment between α and β.

• There exists a smooth, one-to-one mapping from the profiles in $Si(6)$ onto the set of points (α, β, d) where $\alpha \in T_1, \beta \in T_2$, and $d \in [0, 1]$. (Saari [5])

$$(p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ \mathbf{p}_6) \to (\alpha,\ \beta,\ d)$$
$$= \left(\frac{p_1}{d}, \frac{p_3}{d}, \frac{\mathbf{p_5}}{\mathbf{d}}, \frac{p_2}{1-d}, \frac{p_4}{1-d}, \frac{\mathbf{p_6}}{\mathbf{1-d}}, p_1 + p_3 + p_5\right)$$

• As the name suggests, profiles can be represented in the Representation Cube.

Geometry of Pairwise Votes: Representation Cube

- WGAD? (or, the value of this representation ...)
  - α's position in T_1 yields information about dominant odd voters
  - β's position in T_2 yields information about dominant even voters
  - d determines whether odd or even voters are more prevelant
- Can construct examples based on α, β, and d where "nearby" profiles will have specific properties
- Which profiles yield the same pairwise outcome?
  - α-cone and β-cone

Another Take on the Representation Cube

Recall: All profiles can be expressed as $\mathbf{p} = \mathbf{p}_K + \mathbf{p}_B + \mathbf{p}_C + \mathbf{p}_R$ where the profile differentials come from the Kernel, the Basic, the Condorcet, and the Reversal subspaces.

- How do these vectors relate to the pairwise comparisons?

$$\mathbf{R}_A = (1, 1, -2, 1, 1, -2) \;\; \mathbf{R}_B = (-2, 1, 1, -2, 1, 1) \;\; \mathbf{R}_C = (1, 1, -2, 1, 1, -2)$$

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• Saari [7]: only the Basic and Condorcet vectors, $\mathbf{p}_B + \mathbf{p}_C$, affect pairwise outcomes

− What do these vectors/subspaces have to do with the Representation Cube?

Another Take on the Representation Cube

- The Basic subspace forms a plane in the Representation Cube.

  – call this the *transitivity plane*; it is the plane $x_1 + x_2 + x_3 = 0$ and therefore only intersects the transitive regions

- Every pairwise comparison outcome $\mathbf{q}$ can be orthogonally decomposed as a vector in the transitivity plane plus a scalar times the transitivity plane's normal vector.

$$\mathbf{q} = \mathbf{q}_T + \mu(1, 1, 1)$$

  – call $(1, 1, 1)$ the cyclic axis

Another Take on the Representation Cube

• What is the implication of the transitivity plane?

Definition(s) The BC ranking is equivalent to ranking the candidates according to assigned scores where the j^{th} candidate's score is

$$BC(c_j) = \sum_{k \neq j} d_{j,k}.$$

The Copeland ranking is determined by replacing $d_{j,k}$ by 1, 0, or -1 depending on the sign of $d_{j,k}$. Copeland winner has largest sum.

The Kemeny ranking for profile $\mathbf{p}$ is a transitive ranking obtained from $F_3(\mathbf{p})$ by reversing $d_{j,k}$'s so that **1.** the resulting ranking is transitive and **2.** minimize the sum of the reversed terms.

Equivalently, the KR is the nearest (ℓ_1 distance) transitive ranking region to $F_3(\mathbf{p})$.

Theorem All of the procedures defined above have the same election outcome when the profile is a Basic profile.

Applying the Representation Cube: Dodgson's Procedure

• This material is adapted from Ratliff [4].

• Charles Dodgson [1] (in McLean and Urken [2]) describes an election procedure that generalizes the Condorcet winner. Consider the example:

5	2	4	3	4	1
A	A	C	C	B	B
B	C	A	B	C	A
C	B	B	A	A	C

$\Rightarrow$

$A \succ B$	$B \succ A$	$A \succ C$	$C \succ A$	$B \succ C$	$C \succ B$
11	8	8	11	10	9

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• Dodgson winner is the "closest" candidate to the Condorcet winner by minimizing its Dodgson number: the minimum number of pairwise switches for the candidate to become the Condorcet winner

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• Ratliff [4] relates the Dodgson winner to the ℓ_1 metric (taxicab geometry).

– The ℓ_1 winner is the Condorcet winner of the ranking corresponding to the orthant that has the smallest ℓ_1 distance to the point in the pairwise space among all orthants that correspond to a Condorcet winner.

Applying the Representation Cube: Dodgson's Procedure

• Back to the example:

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$A \succ B$	$B \succ A$	$A \succ C$	$C \succ A$	$B \succ C$	$C \succ B$
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• Under the ℓ_1 metric, we see that $(d_{A \succ B}, d_{B \succ C}, d_{C \succ A}) = (3, 1, 3)$ requires a single change.

References

[1] C. Dodgson. A method of taking votes on more than two issues. (1876) In [2].

[2] I. McLean and A. Urken. (editors) *Classics of social choice.* Ann Arbor, MI: University of Michigan Press, 1995.

[3] T. Ratliff. A comparison of Dodgson's method and Borda Count. *Economic Theory* **20** (2002) 357-372.

[4] T. Ratliff. A comparison of Dodgson's method and Kemeny's rule. *Social Choice and Welfare* **18** (2001) 79-89.

[5] D. Saari. *Basic Geometry of Voting.* New York, NY: Springer Verlag, 1995.

[6] D. Saari. *Geometry of Voting.* New York, NY: Springer Verlag, 1994.

[7] D. Saari. Explaining all three-alternative voting outcomes. *Journal of Economic Theory* **87** (1999) 313-355.

[8] D. Saari. Mathematical structure of voting paradoxes. *Economic Theory* **15** (2000) 1-53.