THE SPACE OF POSITIONAL ELECTION OUTCOMES: Applications, Examples, and Techniques

DIMACS Reconnect: Mathematics of Elections and Decisions
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Recall

Assume there are 3 candidates, A, B, and C. For a profile of voters $\mathbf{p}$, a value s defines a voting vector $\begin{bmatrix} 1-s \\ s \\ 0 \end{bmatrix}$. The election outcome is determine by matrix multiplication

$$\begin{bmatrix} 1-s & 1-s & s & 0 & 0 & s \\ s & 0 & 0 & s & 1-s & 1-s \\ 0 & s & 1-s & 1-s & s & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

By varying s from 0 to $\frac{1}{2}$, we get all possible election outcomes for a profile.

An Application of the Geometry of Voting to the 1992 US Presidential Election

Tabarrok [9] used geometry to examine the 1992 US Presidential Election.

Candidate	Actual
Clinton	0.4295
Bush	0.3740
Perot	0.1885

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Clinton	0.4295	0.5062
Bush	0.3740	0.3849
Perot	0.1885	0.1089

Profile	p_1	p_2	p_3	p_4	p_5	p_6
October 1992	0.2085	0.2977	0.0583	0.0506	0.1744	0.2105

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October 1992	0.2085	0.2977	0.0583	0.0506	0.1744	0.2105
Adjusted	0.2085	0.2210	0.1350	0.0615	0.1635	0.2105

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Candidate	**Actual**	**Poll**	**Adjusted**	**Antiplurality**
Clinton	0.4295	0.5062	0.4295	0.3875
Bush	0.3740	0.3849	0.3740	0.3220
Perot	0.1885	0.1089	0.1965	0.2905

Profile	p_1	p_2	p_3	p_4	p_5	p_6
October 1992	0.2085	0.2977	0.0583	0.0506	0.1744	0.2105
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- Clinton is the plurality and antiplurality winner.

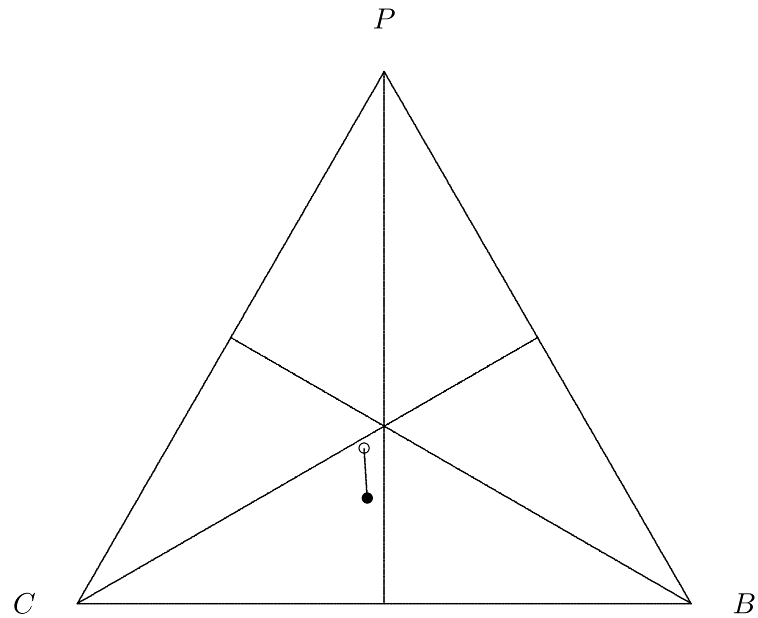

Figure 1: Plurality (•) and Anti-plurality (∘) outcomes for the 1992 US Presidential Election.

An Application of the Geometry of Voting to the 1992 US Presidential Election

• Clinton wins under every scoring vector.

• Other procedures? (*e.g.*, Approval Voting and Cumulative Voting?) And, how do they visually compare?

Determining Approval Voting (AV) Outcomes

A voter may submit a vote for its top choice or its top-two choices. Let r_i be the percentage of voters of type i that vote for their top-two choices.

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p_1	p_2	p_3	p_4	p_5	p_6
C	C	P	P	B	B
B	P	C	B	P	C
P	B	B	C	C	P

$\Rightarrow$

Candidate	"Number of Votes"
C	$p_1 + p_2 + r_3 p_3 + r_6 p_6$
B	$p_5 + p_6 + r_1 p_1 + r_4 p_4$
P	$p_3 + p_4 + r_2 p_2 + r_5 p_5$

- What happens if $r_i = 0$ for all i? What happens it $r_i = 1$ for all i?

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B	P	C	B	P	C
P	B	B	C	C	P

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P	$p_3 + p_4 + r_2 p_2 + r_5 p_5$

• What happens if $r_i = 0$ for all i? What happens it $r_i = 1$ for all i?

• Every outcome from a positional method can be achieved under AV. (Saari and Van Newenhizen [7])

The "Number of Votes" is linear in the r_i's. Again, by convexity, we can determine all outcomes by considering the extreme values for the r_i's, *i.e.*, $r_i \in \{0, 1\}$ for each i. There are 2^6 possibilities.

Determining AV Outcomes

The convex hull defined by six corner points yields all AV outcomes. These are generated by the following r_i values.

r_1	r_2	r_3	r_4	r_5	r_6
1	1	0	1	1	0
0	1	0	0	1	0
0	1	1	0	1	1
0	0	1	0	0	1
1	0	1	1	0	1
1	0	0	1	0	0

- Why these six?
- For example, $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, $r_4 = 1$, $r_5 = 1$, and $r_6 = 0$.
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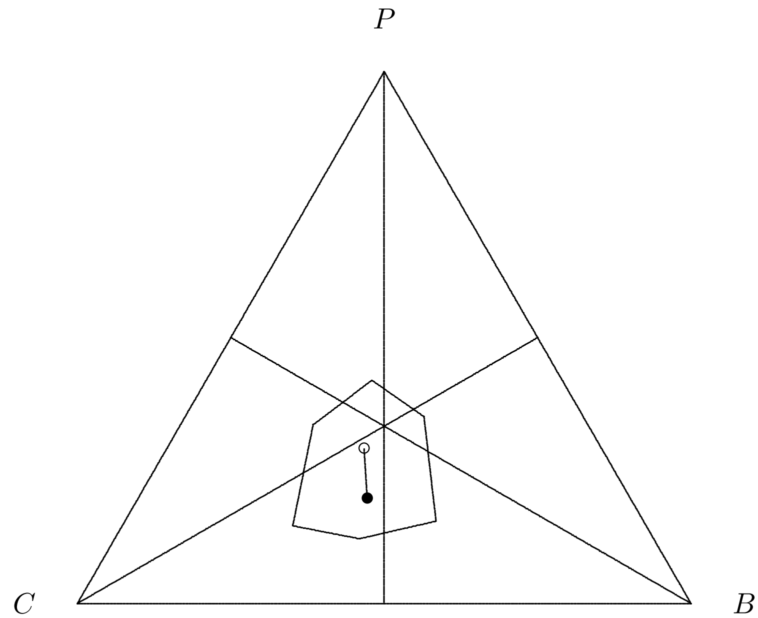

Figure 2: AV outcomes for 1992 US Presidential Election: Interior of hexagonal region.

AV Outcomes: Our Example and In General

- Any of the six rankings is a possible outcome.
- Brams and Fishburn [1] and Brams, Fishburn, and Merrill [2] argue that this flexibility better allows voters to express their preferences.
- Saari and Van Newenhizen [7, 8] disagree; Tabarrok [9] uses the 1992 election to support the Saari and Van Newenhizen view.
- Tabarrok's argues:
  - Profile doesn't determine election outcome.
  - AV is inconsistent with *any* standard of voting based on ordinal voter rankings (*e.g.*, Condorcet winners should never come in last).
  - AV doesn't allow intensity of preference to be represented.

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A voter may split three votes between the three candidates.

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p_1	p_2	p_3	p_4	p_5	p_6
C	C	P	P	B	B
B	P	C	B	P	C
P	B	B	C	C	P

$\Rightarrow$

Candidate	"Number of Votes"
C	$(3 - r_1)p_1 + (3 - r_2)p_2 + r_3 p_3 + r_6 p_6$
B	$(3 - r_5)p_5 + (3 - r_6)p_6 + r_1 p_1 + r_4 p_4$
P	$(3 - r_3)p_3 + (3 - r_4)p_4 + r_2 p_2 + r_5 p_5$

- What happens if $r_i = 0$ for all i? What happens it $r_i = 1$ for all i?

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A voter may split three votes between the three candidates.

p_1	p_2	p_3	p_4	p_5	p_6
C	C	P	P	B	B
B	P	C	B	P	C
P	B	B	C	C	P

$\Rightarrow$

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B	$(3-r_5)p_5 + (3-r_6)p_6 + r_1p_1 + r_4p_4$
P	$(3-r_3)p_3 + (3-r_4)p_4 + r_2p_2 + r_5p_5$

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• Every outcome from a positional method can be achieved under CV. (Saari and Van Newenhizen [7])

The "Number of Votes" is linear in the r_i's. Again, by convexity, we can determine all outcomes by considering the extreme values for the r_i's, *i.e.*, $r_i \in \{0, 1\}$ for each i. There are 2^6 possibilities.

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The convex hull defined by six corner points yields all CV outcomes. These are generated by the following r_i values.

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1	1	0	0	1	0
1	1	0	1	0	0
1	0	1	1	0	0
0	1	0	0	1	1
0	0	1	1	0	1
0	0	1	0	1	1

- Why these six?
- For example, $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, $r_4 = 0$, $r_5 = 1$, and $r_6 = 0$.
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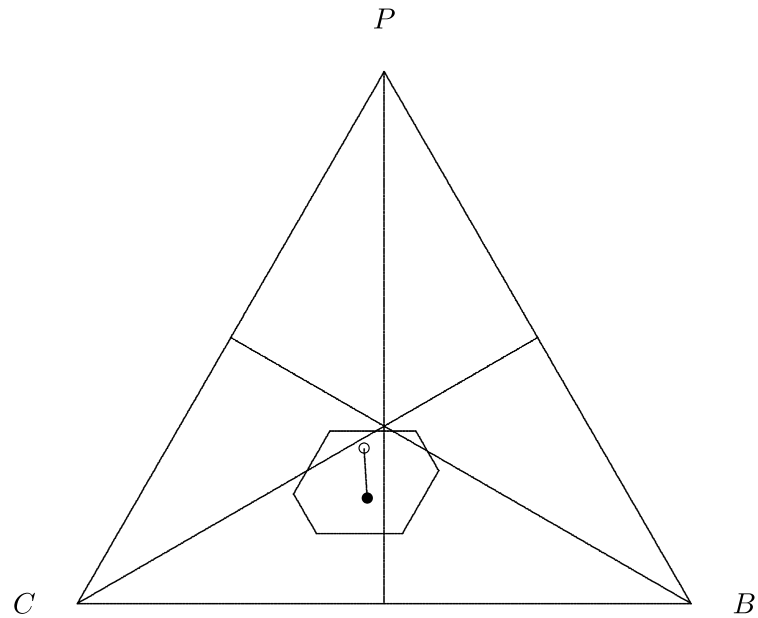


Figure 3: CV outcomes for 1992 US Presidential Election: Interior of hexagonal region.

CV Outcomes: Our Example and In General

• Perot cannot win an election under CV. However, this is not true if CV is used where voters can be indifferent between their top-two choices.

• Similar problems will CV as with AV.

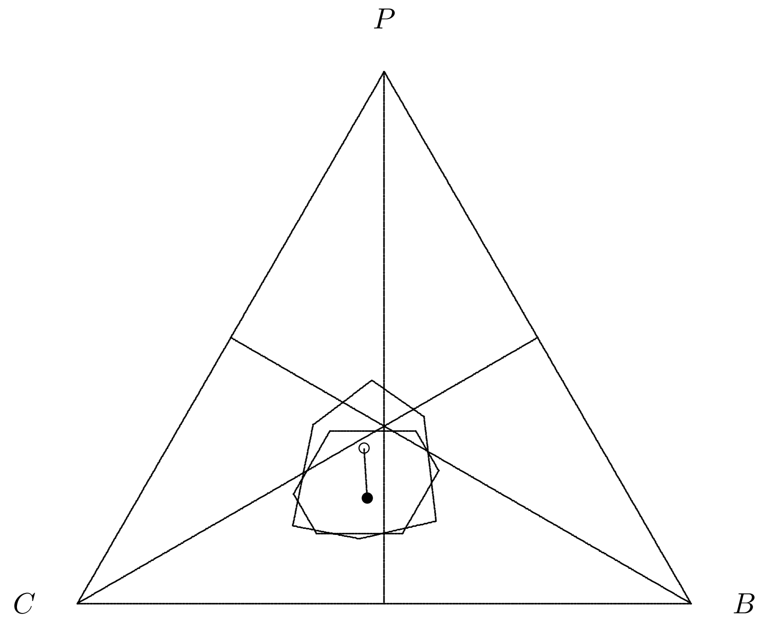


Figure 4: AV and CV outcomes for 1992 US Presidential Election. AV region is “larger” of two regions.

Graphically Constructing Examples

• We can draw a line in the simplex to represent an example. Which lines are possible?

• Not every possible plurality outcome can be paired with every possible antiplurality outcome.

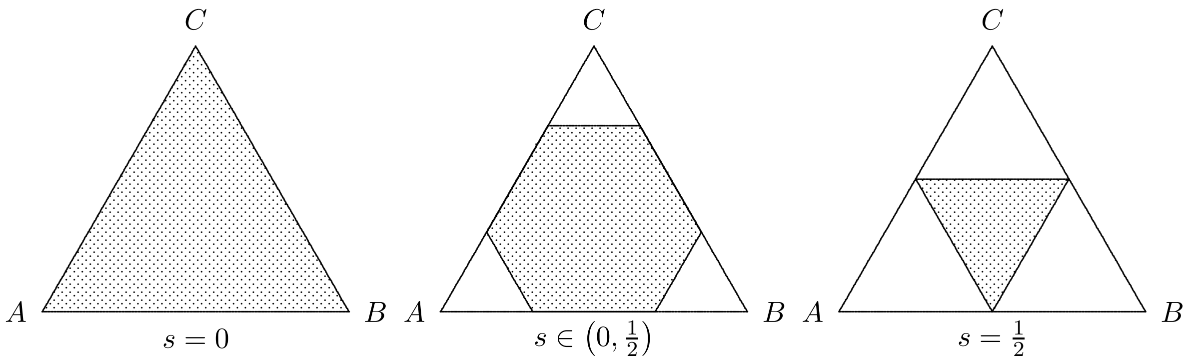


Figure 5: Election outcomes are dependent on the value of $s \in \left[0, \frac{1}{2}\right]$.

Plurality $\Rightarrow$ Possible Antiplurality Outcomes

• Suppose that $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the plurality outcome, then

$$\begin{aligned} p_1 + p_2 &= a \\ p_5 + p_6 &= b \\ p_3 + p_4 &= c \end{aligned}$$

• For $i = 1$ to 3, let $r_i = \dfrac{p_{2i-1}}{p_{2i-1} + p_{2i}}$.

• Again, by a convexity argument, we can consider the 8 possible corner points defined by $r_i \in \{0, 1\}$ for all i.

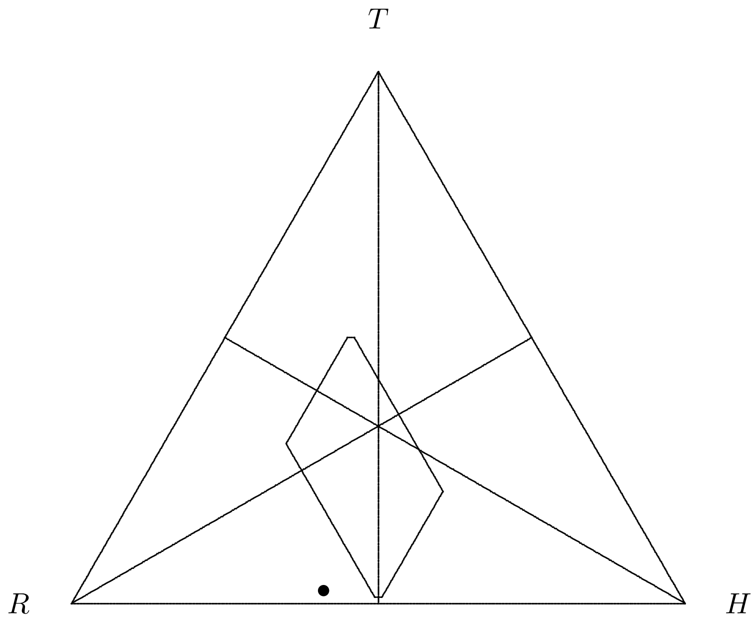


Figure 6: Plurality (•) $\approx [0.578, 0.399, 0.022]^T$ and possible Anti-plurality outcomes for the 1932 (Roosevelt-Hoover-Thomas) US Presidential Election.

Plurality ⇒ Possible Antiplurality Outcomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

- How do d, e, and f relate to a, b, and c?
- Notice: Profiles are in 5 dimensions while plurality and antiplurality (a, b, d, and f) use up on 4 dimensions.
- An infinite number of profiles yield the same plurality outcome and antiplurality outcome.

Geometrical Approach: Example

• Jones [3] looks at geometry when information is incomplete.

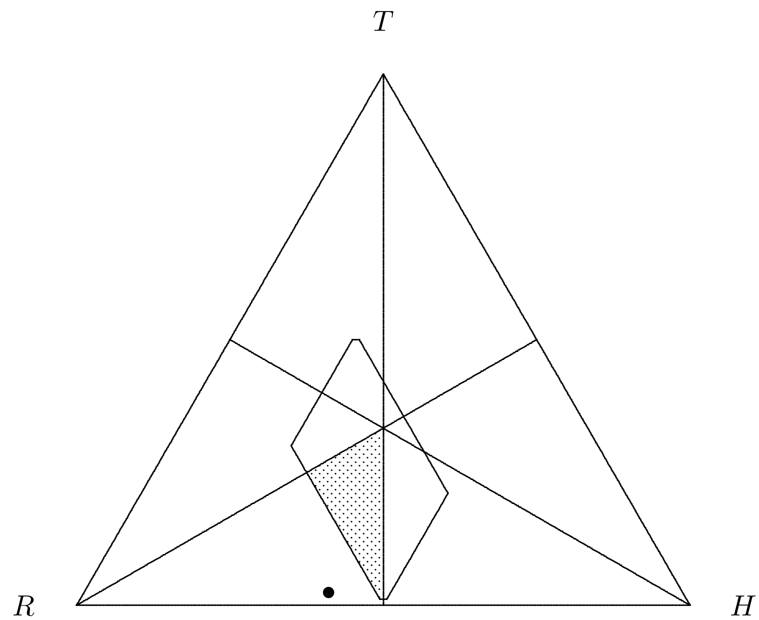


Figure 7: Plurality (•) ≈ $[0.578, 0.399, 0.022]^T$ and possible Anti-plurality outcomes for the 1932 (Roosevelt-Hoover-Thomas) US Presidential Election.

Higher Dimensions?

• Tabarrok and Spector [10] consider the 1860 US Presidential Election with plurality outcome $[0.3979, 0.2939, 0.182, 0.126]^T$ (in the order of Lincoln-L, Douglas-D, Breckinridge-R, and Bell-B)

• Riker [4] estimates the profile as

0.0959^{LDBR} 0.3020^{LBDR} 0.1790^{DLRB} 0.0676^{DLBR} $0.0367^{DRLB}$$0.1037^{DRBL}$ 0.0679^{DBLR} 0.0224^{RDLB} 0.0708^{RDBL} 0.0888^{RBLD} $0.0577^{BLDR}$$0.0244^{BDLR}$ 0.0060^{BDRL} 0.0066^{BRLD} 0.0312^{BRDL}

• Tabarrok and Spector [10] polls historians to arrive at the profile

0.2117^{LDBR} 0.1861^{LBDR} 0.0011^{DLRB} 0.0804^{DLBR} 0.0022^{DRLB} 0.0487^{DRBL} 0.0859^{DBLR} 0.0753^{DBRL} 0.0013^{RDLB} 0.0687^{RDBL} 0.1119^{RBLD} 0.0170^{BLDR} 0.0448^{BDLR} 0.0381^{BDRL} 0.0004^{BRLD} 0.0256^{BRDL}

• All voting vectors $[1 - s_1 - s_2, s_1, s_2, 0]$ where $0 \leq s_2 \leq s_1 \leq \frac{1-s_2}{2}$ form a plane in the 3-dimensional tetrahedronal simplex.

Candidate	**Riker's Profile**	**Survey Profile**
Lincoln	$39.79 + 14.32 + 15.8s_2$	$39.78 + 9.85s_1 + 13.46s_2$
Douglas	$29.41 + 21.95s_1 + 47.97s_2$	$29.36 + 36.46s_1 + 34.06s_2$
Bell	$12.59 + 45.87s_1 + 33.8s_2$	$12.59 + 45.92s_1 + 49.95s_2$
Breckenridge	$18.2 + 17.82s_1 + 2.39s_2$	$18.19 + 7.69s_1 + 11.45s_2$

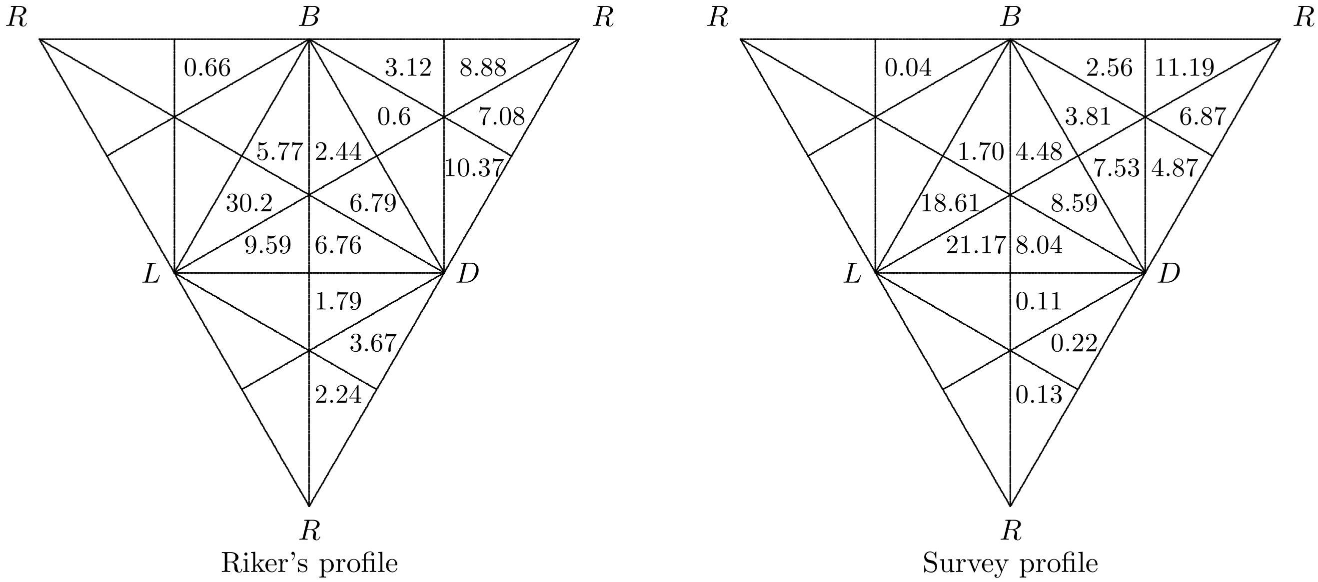
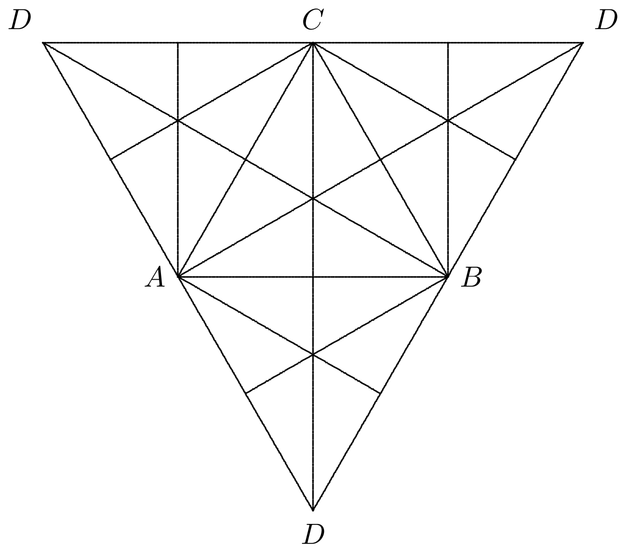


Figure 8: Flattened Tetrahedron: Representing the 1860 US Presidential Election (Saari [6]).

Constructing Examples with 4 Candidates



A suggestion in Saari [6] is to construct an example under plurality rule where

- $A \succ B \approx C \succ D$ with a ratio of 7:6:6:5,
- $C \succ B \succ A$ when D drops out,
- $B \succ C$ when A drops out, and
- cyclic pairwise outcomes of $A \succ B$, $B \succ C$, $C \succ A$.

References

[1] S.J. Brams and P.C. Fishburn. *Approval Voting.* Boston, MA: Birkhauser, 1983.

[2] S.J. Brams, P.C. Fishburn, and S. Merrill, III. The responsiveness of approval voting: Comments on Saari and Van Newenhizen. *Public Choice* **59** (1988) 121-131.

[3] M. Jones. Applying the geometry of voting when information is incomplete. (2005) In progress.

[4] W. Riker. *Liberalism against Populism: A Confrontation between the Theory of Deomocracy and the Theory of Social Choice.* San Francisco, CA: W.H. Freeman, 1982.

[5] D. Saari. *Geometry of Voting.* New York, NY: Springer Verlag, 1994.

[6] D. Saari. *Chaotic Elections! A Mathematician Looks at Voting.* Providence, RI: American Mathematical Society, 2001.

[7] D. Saari and J. van Newenhizen. The problem of indeterminacy in approval, multiple, and truncated voting systems. *Public Choice* **59** (1988) 101-120.

[8] D. Saari and J. van Newenhizen. Is approval voting an unmitigated evil?: A response to Brams, Fishburn, and Merrill. *Public Choice* **59** (1988) 133-147.

[9] A. Tabarrok. President Perot or fundamentals of voting theory illustrated with the 1992 election *Public Choice* **106** (2001) 275-297.

[10] A. Tabarrok and L. Spector. Would the Borda Count have avoided the Civil War? *Journal of Theoretical Politics* **11**, no. 2 (1999) 261-288.