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ON THE 1D AND 2D ROGERS–RAMANUJAN CONTINUED FRACTIONS*

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In this paper the classical and generalized numerical Rogers−Ramanujan continued fractions are extended to a polynomial continued fraction in one and two dimensions. Using the new continued fractions, the fundamental recurrence formulas and a fast algorithm, based on matrix formulations, are given for the computation of their transfer functions. The presented matrix formulations can provide a new perspective to the analysis and design of Ladder-continued fraction filters in one and two dimensions signal processing. The simplicity and efficiency of the presented algorithms are illustrated by step-by-step examples.

Keywords: Rogers−Ramanujan; continued fractions; inversion; 1D/2D systems; multidimensional systems.

1. Introduction

Continued fraction expansions (CFE) have been studied for many centuries since the series of quotients in Euclid's greater common divisor algorithm generated a continued fraction expansion.[1,2] After a long time period, CFE theoretical ideas and algorithmic methods were transferred to the engineering field. CFE have been used extensively in the areas of classical network theory, signal analysis/processing and control systems. Especially, CFE have been applied in digital filtering having lattice and ladder structures, in feedback control system design, in the circuit and state space realization of regular and multidimensional filters and systems.[3−10]

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In literature there exists many types of CFE with different structural characteristics and behavior. In the realization theory area of two dimensional (2D) systems, CFE play an important role because it has been shown that in the case where a 2D transfer function is expanded into any type of CFE, it can be realized with a minimum number of delay elements.[5,6,8,9] Today the minimality is not related to the financial cost of the memory space, but to the execution speed of the filter.

In this paper the classical and the generalized Rogers−Ramanujan CFE types,[11−15] are extended to one and two dimensions, in the polynomial sense. The new polynomial CFE have unique structures, with the variables to be of increasing order. As the order of the CFE increase, the degree of the complexity increases rapidly. In addition, in this paper, the matrix fundamental formula for inverting CFE is modified and applied for the inversion of the proposed one dimensional (1D) and 2D CFE structures. The fundamental recurrence formulas, that also is used in the paper, and Routh's algorithm are other methods, among many, for inverting continued fractions.[16,17]

The presented structure utilizes an analysis based on matrix formulations that provides a new insight to a continued fraction framework in the ladder filtering design. It would be possible, using the new matrix formulations, to modify a continued fraction-ladder based filter structure in the direction of minimizing the coefficient sensitivity and dealing with the roundoff effects.[18] It is noted that not all 1D Ladder structures based on continued-ladder expansions have been efficient with respect to signal processing issues like sensitivity and roundoff error, in many cases 1D digital Ladder structures can be implemented with lower coefficient word lengths than the conventional structures.[19]

2. Rogers–Ramanujan CFE

The "classical" Rogers−Ramanujan continued fraction expansion has the following form[11,12]:

$$T(1,q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{\cdots}}}}\,, \tag{1}$$

where $|q| < 1$.

Recently, Berndt-Yee proposed the "generalized Rogers−Ramanujan continued fraction"[14]:

$$T(a,q) = \cfrac{1}{1 + \cfrac{aq}{1 + \cfrac{aq^2}{1 + \cfrac{aq^3}{\cdots}}}}\,, \tag{2}$$

where $|q| < 1$, and a is a complex number.

3. 1D and 2D Polynomial CFE

3.1. *1D polynomial CFE*

Two symmetric generalized structures of the classical Rogers−Ramanujan CFE,[15] and the generalized Rogers−Ramanujan CFE,[14] extended to polynomial 1D, in the frequency-domain, are expressed as follows,

$$T_A(z) = 1 + \cfrac{a_1 z}{1 + \cfrac{a_2 z^2}{1 + \cfrac{a_3 z^3}{1 + \cfrac{a_4 z^4}{\ddots \over 1 + \cfrac{a_{n-1} z^{n-1}}{1 + a_n z^n}}}}} \tag{3}$$

and

$$T_B(z) = \cfrac{1}{1 + \cfrac{a_1 z}{1 + \cfrac{a_2 z^2}{1 + \cfrac{a_3 z^3}{\ddots \over 1 + \cfrac{a_{n-1} z^{n-1}}{1 + a_n z^n}}}}}\,, \tag{4}$$

where $\{a_1, a_2, \ldots, a_n\}$ are real coefficients of the CFE, and z, is complex (delay element). It is noted that $z = Ae^{j\phi}$, A is the magnitude of z, and ϕ is the angle.[20] The CFE $T_B(z)$ is a natural extension to the polynomial 1D of the generalized Rogers−Ramanujan continued fraction (2) and of the "classical" Rogers−Ramanujan CFE (1). The 1D CFE $T_A(z)$, Eq. (3), can be considered as the generalized 1D continued fraction of the Rogers−Ramanujan type. It is noted that the generalized 2D CFE $T_A(z)$ (3) will be considered throughout this study, except for two examples that the $T_B(z)$ CFE used.

Using the fundamental recurrence formulas (FRF) to invert the continued fraction (3), yields.[21]

Therefore the transfer function $T_A(z)$, using the results of the above Table 1, is defined as,

$$T_A(z) = \frac{N_{T_A}(z)}{D_{T_A}(z)}\,, \tag{5}$$

where $N_{T_A}(z)$ is the numerator polynomial that is equal to $N_{n+1}(z)$ and $D_{T_A}(z)$ is the denominator polynomial that is equal to $D_{n+1}(z)$. For simplicity this notation will be used throughout the paper.

Table 1. Fundamental recurrence formulas for Eq. (3).

Numerator of $T(z)$	Denominator of $T(z)$
$N_0(z) = 1$	$D_0(z) = 1$
$N_1(z) = 1 + a_1 z$	$D_1(z) = 1$
$N_{n+1}(z) = N_n + a_{n+1} z^{n+1} N_{n-1}$	$D_{n+1}(z) = D_N + a_{n+1} z^{n+1} D_{n-1}$

3.1.1. *1D example*

Consider the following low-order 1D CFE:

$$T_1(z) = 1 + \cfrac{b_1 z}{1 + \cfrac{b_2 z^2}{1 + b_3 z^3}} . \tag{6}$$

Using the FRF yields,

Numerator of $T_1(z)$	Denominator of $T_1(z)$
$N_0(z) = 1$	$D_0(z) = 1$
$N_1(z) = 1 + b_1 z$	$D_1(z) = 1$
$N_2(z) = 1 + b_1 z + b_2 z^2$	$D_2(z) = 1 + b_2 z^2$
$N_3(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 (1 + b_1 z)$	$D_3(z) = 1 + b_2 z^2 + b_3 z^3$

Therefore the transfer function $T_1(z)$, of (6) is:

$$T_1(z) = \frac{N_{T_1}(z)}{D_{T_1}(z)} , \tag{7}$$

where

$$\begin{aligned} N_{T_1}(z) &= 1 + b_1 z + b_2 z^2 + b_3 z^3 + b_1 b_3 z^4 \\ D_{T_1}(z) &= 1 + b_2 z^2 + b_3 z^3 \end{aligned} ,$$

with $N_{T_1}(z) = N_3(z)$ and $D_{T_1}(z) = D_3(z)$.

3.2. *2D polynomial CFE*

The 1D CFE (3), extended to two dimensions (2D), takes the following form:

$$T_2(z_1, z_2) = 1 + \cfrac{a_1 z_1}{1 + \cfrac{b_1 z_2}{1 + \cfrac{a_2 z_1^2}{1 + \cfrac{b_2 z_2^2}{\cfrac{\ddots}{1 + \cfrac{a_n z_1^n}{1 + b_n z_2^n}}}}}} . \tag{8}$$

It is noted that z_1, z_2 are delay elements.

Inverting the above CFE (8), using the fundamental recurrence formulas, the resulting 2D transfer function will have the following form:

$$T_2(z_1, z_2) = \frac{N_{T_2}(z_1, z_2)}{D_{T_2}(z_1, z_2)}\,. \tag{9}$$

3.2.1. *2D example*

Consider the following low-order 2D CFE:

$$T_3(z_1, z_2) = 1 + \cfrac{a_1 z_1}{1 + \cfrac{b_1 z_2}{1 + \cfrac{a_2 z_1^2}{1 + b_2 z_2^2}}}\,. \tag{10}$$

Using the FRF, to invert the continued fraction (10), the 2D numerator and denominator polynomials are:

Numerator of $T_3(z_1, z_2)$
$N_0(z_1, z_2) = 1$
$N_1(z_1, z_2) = 1 + a_1 z_1$
$N_2(z_1, z_2) = 1 + a_1 z_1 + b_1 z_2$
$N_3(z_1, z_2) = 1 + a_1 z_1 + b_1 z_2 + a_2 z_1^2(1 + a_1 z_1)$
$N_4(z_1, z_2) = 1 + a_1 z_1 + b_1 z_2 + a_2 z_1^2(1 + a_1 z_1) + (1 + a_1 z_1 + b_1 z_2) b_2 z_2^2$

and

Denominator of $T_3(z_1, z_2)$
$D_0(z_1, z_2) = 1$
$D_1(z_1, z_2) = 1$
$D_2(z_1, z_2) = 1 + b_1 z_2$
$D_3(z_1, z_2) = 1 + b_1 z_2 + a_2 z_1^2$
$D_4(z_1, z_2) = 1 + b_1 z_2 + a_2 z_1^2 + b_2 z_2^2(1 + b_1 z_2)$

Therefore the transfer function $T_3(z_1, z_2)$, of (10) is:

$$T_3(z_1, z_2) = \frac{N_{T_3}(z_1, z_2)}{D_{T_3}(z_1, z_2)}\,, \tag{11}$$

where

$$\begin{aligned} N_{T_3}(z_1, z_2) &= 1 + a_1 z_1 + b_1 z_2 + a_2 z_1^2 + a_1 a_2 z_1^3 + b_2 z_2^2 + a_1 b_2 z_1 z_2^2 + b_1 b_2 z_2^3\,, \\ D_{T_3}(z_1, z_2) &= 1 + b_1 z_2 + a_2 z_1^2 + b_2 z_2^2 + b_1 b_2 z_2^3\,. \end{aligned}$$

It is noted that due to the nature of the Rogers–Ramanujan continued fractions, some terms in the corresponding 1D and 2D transfer function are missing.

4. 1D and 2D CFE Inversion

In the following subsections the modified inversion matrix formulations presented for the proposed 1D and 2D CFE of Rogers–Ramanujan types.

4.1. *1D matrix inversion formula*

Using the independent results of matrix formulation for continued fraction inversion,[21–25] the transfer function (5) of the 1D continued fraction (3) can be determined as follows:

$$\begin{bmatrix} N(z) \\ D(z) \end{bmatrix} = \mathbf{P}(\mathbf{Q})\mathbf{r}_{n,z^n}\,, \tag{12}$$

where $\mathbf{Q} = \mathbf{Q}_{1,z}\mathbf{Q}_{2,z^2}\mathbf{Q}_{3,z^3},\ldots,\mathbf{Q}_{n-1,z^{n-1}}$.

The dimensions of the matrices $\mathbf{P}, \mathbf{Q}_{1,z}, \mathbf{Q}_{2,z^2}, \mathbf{Q}_{3,z^3}, \mathbf{Q}_{n-1z^{n-1}}$ are (2×2), and of the vector $\mathbf{r}_{n,z^n}$ is (2×1), having the following structures:

$$\begin{aligned}
\mathbf{P} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \\
\mathbf{Q}_{1,z} &= \begin{bmatrix} 1 & 1 \\ a_1 z & 0 \end{bmatrix}, \quad \mathbf{Q}_{2,z^2} = \begin{bmatrix} 1 & 1 \\ a_2 z_2 & 0 \end{bmatrix}, \\
\mathbf{Q}_{3,z^3} &= \begin{bmatrix} 1 & 1 \\ a_3 z^3 & 0 \end{bmatrix}, \quad \mathbf{Q}_{n-1,z^{n-1}} = \begin{bmatrix} 1 & 1 \\ a_{n-1z^{n-1}} & 0 \end{bmatrix}, \\
\mathbf{r}_{n,z^n} &= \begin{bmatrix} 1 \\ a_n z^n \end{bmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} N(z) \\ D(z) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ a_1 z & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ a_2 z^2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ a_3 z^3 & 0 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 1 \\ a_4 z^4 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ a_{n-1} z^{n-1} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ a_n z_n \end{bmatrix}.
\end{aligned} \tag{13}$$

For faster calculation the matrix multiplication should start from right-to-left.[23]

4.1.1. *1D numerical example-1*

Consider the following polynomial 1D CFE:

$$T_4(z) = 1 + \cfrac{z}{1 + \cfrac{2z^2}{1 + 3z^3}}\,. \tag{14}$$

Using the FRF, to invert the continued fraction (14), yields:

Numerator of $T_4(z)$	Denominator of $T_4(z)$
$N_0(z) = 1$	$D_0(z) = 1$
$N_1(z) = 1 + z$	$D_1(z) = 1$
$N_2(z) = 1 + z + 2z^2$	$D_2(z) = 1 + 2z^2$
$N_3(z) = 1 + z + 2z^2 + 3z^3(1 + z)$	$D_3(z) = 1 + 2z^2 + 3z^3$

Therefore the transfer function $T_4(z)$ of (14) is

$$T_4(z) = \frac{N_{T_4}(z)}{D_{T_4}(z)}\,, \tag{15}$$

where

$$\begin{aligned} N_{T_4}(z) &= 1 + z + 2z^2 + 3z^3 + 3z^4\,, \\ D_{T_4}(z) &= 1 + 2z^2 + 3z^3\,. \end{aligned}$$

Using the 1D matrix inversion formula (12)

$$\begin{aligned} \begin{bmatrix} N_{T_4}(z) \\ D_{T_4}(z) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_1 z & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_2 z^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z^2}} \underbrace{\begin{bmatrix} 1 \\ a_3 z^3 \end{bmatrix}}_{\mathbf{r}_{3,z^3}} \\ &= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 1 & 1 \\ z & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z}} \underbrace{\begin{bmatrix} 1 & 1 \\ 2z^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z^2}} \underbrace{\begin{bmatrix} 1 \\ 3z^3 \end{bmatrix}}_{\mathbf{r}_{3,z^3}} \\ &= \begin{bmatrix} 1 + z + 2z^2 + 3z^3 + 3z^4 \\ 1 + 2z^2 + 3z^3 \end{bmatrix}. \end{aligned} \tag{16}$$

The resulting vector polynomial (16) verifies the 1D transfer function result given in Eq. (15).

4.1.2. *1D numerical example-2*

Consider the following polynomial CFE, which is the extension to 1D of the "generalized Rogers–Ramanujan" CFE, given in Eqs. (2) and (4):

$$T_5(z) = \cfrac{1}{1 + \cfrac{2z}{1 + 3z^2}}\,. \tag{17}$$

Using the FRF, to invert the above continued fraction (17), yields:

$$T_5(z) = \frac{N_{T_5}(z)}{D_{T_5}(z)}\,, \tag{18}$$

where

$$N_{T_5}(z) = 1 + 3z^2\,,$$
$$D_{T_5}(z) = 1 + 2z + 3z^2\,.$$

Since Eq. (17) has the structure of the $T_B(z)$, Eq. (4), the first two matrices of the 2D inversion formula (12) should be replaced by,

$$\mathbf{P}' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} N_{T_5}(z) \\ D_{T_5}(z) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{P}'} \underbrace{\begin{bmatrix} 1 & 1 \\ a_1 z & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z}} \underbrace{\begin{bmatrix} 1 \\ a_2 z^2 \end{bmatrix}}_{\mathbf{r}_{2,z^2}} \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{P}'} \underbrace{\begin{bmatrix} 1 & 1 \\ 2z & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z}} \underbrace{\begin{bmatrix} 1 \\ 3z^2 \end{bmatrix}}_{\mathbf{r}_{2,z^2}} \\ &= \begin{bmatrix} 1 + 3z^2 \\ 1 + 2z + 3z^2 \end{bmatrix}. \end{aligned} \tag{19}$$

The resulting vector polynomial (19) verifies the 1D transfer function result given in Eq. (18).

4.2. *2D matrix inversion formula*

Using the independent results of matrix formulation for continued fraction inversion,[21–25] the transfer function (9) of the 2D continued fraction (8) can be determined as follows:

$$\begin{bmatrix} N(z_1, z_2) \\ D(z_1, z_2) \end{bmatrix} = \mathbf{P}(\mathbf{Q})\mathbf{r}_{n,z_2^n}\,, \tag{20}$$

where $\mathbf{Q} = \mathbf{Q}_{1,z_1}\mathbf{Q}_{1,z_2}\mathbf{Q}_{2,z_1^2}\mathbf{Q}_{2,z_2^2}, \ldots, \mathbf{Q}_{n,z_1^n}$.

Note that the dimensions of the matrices $\mathbf{P}$, $\mathbf{Q}_{1,z_1}$, $\mathbf{Q}_{1,z_2}$, $\mathbf{Q}_{2,z_1^2}$, $\mathbf{Q}_{2,z_2^2}$, $\mathbf{Q}_{n,z_1^n}$ are (2×2), and of the vector $\mathbf{r}_{n,z_2^n}$, is (2×1), having the following structures:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\mathbf{Q}_{1,z_1} = \begin{bmatrix} 1 & 1 \\ a_1 z_1 & 0 \end{bmatrix}, \quad \mathbf{Q}_{1,z_2} = \begin{bmatrix} 1 & 1 \\ b_1 z_2 & 0 \end{bmatrix},$$
$$\mathbf{Q}_{2,z_1^2} = \begin{bmatrix} 1 & 1 \\ a_2 z_1^2 & 0 \end{bmatrix}, \quad \mathbf{Q}_{2,z_2^2} = \begin{bmatrix} 1 & 1 \\ b_2 z_2^2 & 0 \end{bmatrix},$$

$$\mathbf{Q}_{n,z_1^n} = \begin{bmatrix} 1 & 1 \\ a_n z_1^n & 0 \end{bmatrix},$$
$$\mathbf{r}_{n,z_2^n} = \begin{bmatrix} 1 \\ b_n z_2^n \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} N(z_1,z_2) \\ D(z_1,z_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a_1 z_1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ b_1 z_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a_2 z_1^2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ b_2 z_2^2 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 1 \\ a_n z_1^n & 0 \end{bmatrix} \begin{bmatrix} 1 \\ b_n z_2^n \end{bmatrix}. \tag{21}$$

For faster calculation, the matrix multiplication should start from right to left.[25]

4.2.1. *2D numerical example-1*

Consider the following polynomial 2D CFE:

$$T_6(z_1,z_2) = 1 + \cfrac{z_1}{1 + \cfrac{2z_2}{1 + \cfrac{3z_1^2}{1 + 4z_2^2}}}. \tag{22}$$

Using the FRF to invert the continued fraction (22), the numerator and denominator polynomials are:

Numerator of $T_6(z_1,z_2)$
$N_0(z_1,z_2) = 1$
$N_1(z_1,z_2) = 1 + z_1$
$N_2(z_1,z_2) = 1 + z_1 + 2z_2$
$N_3(z_1,z_2) = 1 + z_1 + 2z_2 + 3z_1^2(1 + z_1)$
$N_4(z_1,z_2) = 1 + z_1 + 2z_2 + 3z_1^2(1 + z_1) + 4z_2^2(1 + z_1 + 2z_2)$

and

Denominator of $T_6(z_1,z_2)$
$D_0(z_1,z_2) = 1$
$D_1(z_1,z_2) = 1$
$D_2(z_1,z_2) = 1 + 2z_2$
$D_3(z_1,z_2) = 1 + 2z_2 + 3z_1^2$
$D_4(z_1,z_2) = 1 + 2z_2 + 3z_1^2 + 4z_2^2(1 + 2z_2)$

Therefore the transfer function $T_6(z_1, z_2)$ of Eq. (22) is

$$T_6(z_1, z_2) = \frac{N_{T_6}(z_1, z_2)}{D_{T_6}(z_1, z_2)}\,, \tag{23}$$

where

$$\begin{aligned} N_{T_6}(z_1, z_2) &= 1 + z_1 + 2z_2 + 3z_1^2 + 3z_1^3 + 4z_2^2 + 4z_1 z_2^2 + 8z_2^3\,, \\ D_{T_6}(z_1, z_2) &= 1 + 2z_2 + 3z_1^2 + 4z_2^2 + 8z_2^3\,. \end{aligned}$$

Applying the 2D matrix inversion formula, given in Eq. (20), yields

$$\begin{bmatrix} N_{T_6}(z_1, z_2) \\ D_{T_6}(z_1, z_2) \end{bmatrix} = \mathbf{P}(\mathbf{Q}_{1,z_1}\mathbf{Q}_{1,z_2}\mathbf{Q}_{2,z_1^2}\mathbf{Q}_{2,z_2^2})\mathbf{r}_{2,z_2^2}$$

or

$$\begin{bmatrix} N_{T_6}(z_1, z_2) \\ D_{T_6}(z_1, z_2) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_1 z_1 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_1}} \underbrace{\begin{bmatrix} 1 & 1 \\ b_1 z_2 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_2}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_2 z_1^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z_1^2}} \times \underbrace{\begin{bmatrix} 1 \\ b_2 z_2^2 \end{bmatrix}}_{\mathbf{r}_{2,z_2^2}}$$

or

$$\begin{bmatrix} N_{T_6}(z_1, z_2) \\ D_{T_6}(z_1, z_2) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 1 & 1 \\ z_1 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_1}} \underbrace{\begin{bmatrix} 1 & 1 \\ 2z_2 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_2}} \underbrace{\begin{bmatrix} 1 & 1 \\ 3z_1^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z_1^2}} \underbrace{\begin{bmatrix} 1 \\ 4z_2^2 \end{bmatrix}}_{\mathbf{r}_{2,z_2^2}}$$

or

$$\begin{bmatrix} N_{T_6}(z_1, z_2) \\ D_{T_6}(z_1, z_2) \end{bmatrix} = \begin{bmatrix} 1 + z_1 + 2z_2 + 3z_1^2 + 3z_1^3 + 4z_2^2 + 4z_1 z_2^2 + 8z_2^3 \\ 1 + 2z_2 + 3z_1^2 + 4z_2^2 + 8z_2^3 \end{bmatrix}. \tag{24}$$

The resulting vector polynomial (24) verifies the 2D transfer function result given in Eq. (23).

4.2.2. *Numerical example-2*

Consider the following polynomial CFE, which is an extension to 2D of the 1D "generalized Rogers−Ramanujan" CFE, given in Eqs. (2) and (4),

$$T_7(z_1, z_2) = \cfrac{1}{1 + \cfrac{z_1}{1 + \cfrac{2z_2}{1 + \cfrac{3z_1^2}{1 + 4z_2^2}}}}\,. \tag{25}$$

Inverting, the 2D transfer function (25), yields

$$\begin{aligned} T_7(z_1,z_2) &= \frac{N_7(z_1,z_2)}{D_7(z_1,z_2)} \\ &= \frac{1+2z_2+3z_1^2+4z_2^2+8z_2^3}{1+z_1+3z_1^3+4z_1z_2^2+3z_1^2+4z_2^2+2z_2+8z_2^3}. \end{aligned} \tag{26}$$

Since the transfer function $T_7(z_1,z_2)$, Eq. (25), has the structure of the $T_B(z)$, Eq. (4), extended to 2D, the first two matrices of the inversion formula (20) should be replaced by,

$$\mathbf{P}' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} N_{T_7}(z_1,z_2) \\ D_{T_7}(z_1,z_2) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{P}'} \underbrace{\begin{bmatrix} 1 & 1 \\ a_1 z_1 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_1}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_2 z_2 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_2}} \underbrace{\begin{bmatrix} 1 & 1 \\ a_3 z_1^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z_1^2}} \\ &\quad \times \underbrace{\begin{bmatrix} 1 \\ a_4 z_2^2 \end{bmatrix}}_{\mathbf{r}_{2,z_2^2}} \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{P}'} \underbrace{\begin{bmatrix} 1 & 1 \\ z_1 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_1}} \underbrace{\begin{bmatrix} 1 & 1 \\ 2z_2 & 0 \end{bmatrix}}_{\mathbf{Q}_{1,z_2}} \underbrace{\begin{bmatrix} 1 & 1 \\ 3z_1^2 & 0 \end{bmatrix}}_{\mathbf{Q}_{2,z_1^2}} \\ &\quad \times \underbrace{\begin{bmatrix} 1 \\ 4z_2^2 \end{bmatrix}}_{\mathbf{r}_{2,z_2^2}}. \end{aligned}$$

Finally,

$$\begin{bmatrix} N_{T_7}(z_1,z_2) \\ D_{T_7}(z_1,z_2) \end{bmatrix} = \begin{bmatrix} 1+2z_2+3z_1^2+4z_2^2+8z_2^3 \\ 1+z_1+3z_1^3+4z_1z_2^2+3z_1^2+4z_2^2+2z_2+8z_2^3 \end{bmatrix}. \tag{27}$$

The resulting vector polynomial (27) verifies the 2D transfer function result given in Eq. (25).

5. Conclusions

In this paper the well known Rogers–Ramanujan CFE were considered. Continued fraction expansions have been studied extensively by mathematicians and engineers for many decades and have found applications in many areas of engineering and sciences. The purpose of this paper is two-fold. First, the numerical "classical" and "generalized Rogers–Ramanujan" CFE were extended to polynomial 1D and 2D continued fractions. The new proposed structures are aesthetically elegant, containing fascinating and surprising results. In the 2D case, the delay elements z_1 and z_2 appear having an alternating ladder structure. Secondly, a fast algorithm based on matrix formulations along with the fundamental recurrence formulas, was presented for the inversion of the continued fractions. It is noted that the results of this paper can be easily extended to higher dimensions. The matrix formulations of the presented continued fraction expansion may offer a new perspective to the further study of Ladder-continued fraction filters in different aspects of signal processing analysis.

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