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4D FIR Digital Filter Realizations

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***Abstract*—This paper proposes generalized circuit and state space realizations for four–dimensional (4D) Finite Impulse Response (FIR) filters. Specifically, lattice and direct-form filter structures are considered. The 4D circuit realizations utilize, for their implementation, a minimum number of delay elements. Further, the dimensions of the state–space vector, of the formulated 4D state space models, are minimal. Examples are given to demonstrate the minimality of their circuit and state–space realizations.**

I. Introduction

Multidimensional filters and systems have been an elegant field in applied science and engineering for a number of years [1]-[3]. The motivation in pursuing the study of systems, and filters with higher dimensions is steadily increasing [1],[4]–[8]. However there are inherent, yet unresolved, challenges yielding unpredictable but interesting results. In spite of this, there have recently been several encouraging publications for 4D filters and systems. The new developing area of research, besides the promising wide variety of possible applications, has delineated notable contributions in 4D depth filtering, first–order 4D infinite impulse response (IIR) frequency–planar digital filters, first–order 4D IIR frequency-hyperplanar, 4D lattice filter realization, etc. [5]–[8].

In this paper generalized circuits and minimal state–space realizations, for 4D FIR lattice and direct–form digital filters, are presented. The necessity to provide minimal realization arises not only out of hardware requirements but also because sometimes non–minimal realizations often cause theoretical or computational difficulties due to the absence of the fundamental theorem of algebra for polynomials with more than one–dimension [2],[3].

Generalized minimal circuit and state space realizations have been reported for 4D lattice structured digital filters [4]. Moreover 4D first-order IIR frequency–planar, and IIR frequency–hyperplanar, 4D frequency–hyperfan have been reported in [5]–[8].

II. Problem Statement

The following 4D state–space equations comprise a cyclic 4D structured model, with respect to the four independent variables: $x_1^h(i,j,k,l), x_1^v(i,j,k,l), x_1^d(i,j,k,l), x_1^t(i,j,k,l)$ [9].

$$\dot{\mathbf{x}}(i,j,k,l) = \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (1)$$

$$y(i,j,k,l) = \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (2)$$

where,

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l)) \\ x_1^t(i,j,k,l) \\ \cdots \\ x_{4n}^h(i,j,k,l) \\ x_{4n}^v(i,j,k,l) \\ x_{4n}^d(i,j,k,l)) \\ x_{4n}^t(i,j,k,l) \end{bmatrix} \quad (3)$$

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l)) \\ x_1^t(i,j,k,l+1) \\ \cdots \\ x_{4n}^h(i+1,j,k,l) \\ x_{4n}^v(i,j+1,k,l) \\ x_{4n}^d(i,j,k+1,l)) \\ x_{4n}^t(i,j,k,l+1) \end{bmatrix} \quad (4)$$

The matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}', \mathbf{d}$ of the above 4D state space model, having the following dimensions respectively: $4n \times 4n$, $4n \times 1$, $1 \times 4n$.

The above model (1,2) can be used to represent in state–space 4D digital filters and systems.

Applying the 4D z–transform on (1,2), its corresponding 4D transfer-function takes the following form:

$$H(z_1, z_2, z_3, z_4) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b} \quad (5)$$

where, $\mathcal{Z} = z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n \oplus z_3\mathbf{I}_n \oplus z_4\mathbf{I}_n$ with $\oplus$ denoting the direct sum.

In the following sections, generalized circuit and state space realizations are presented for FIR lattice and direct–form filters having a minimum number of delay elements and dimension of the state vector.

III. 4D FIR Lattice discrete filters

A generalized 4D FIR digital filter with lattice structure is depicted in Fig. 1., following on the theory of 1D and 2D lattice filters [10],[11].

The proposed generalized circuit utilizes $4n$ delay elements, $8n$ multipliers and adders as shown in Fig. 1., where the number of both components are minimal. $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ denote the multiplier and adder, respectively.

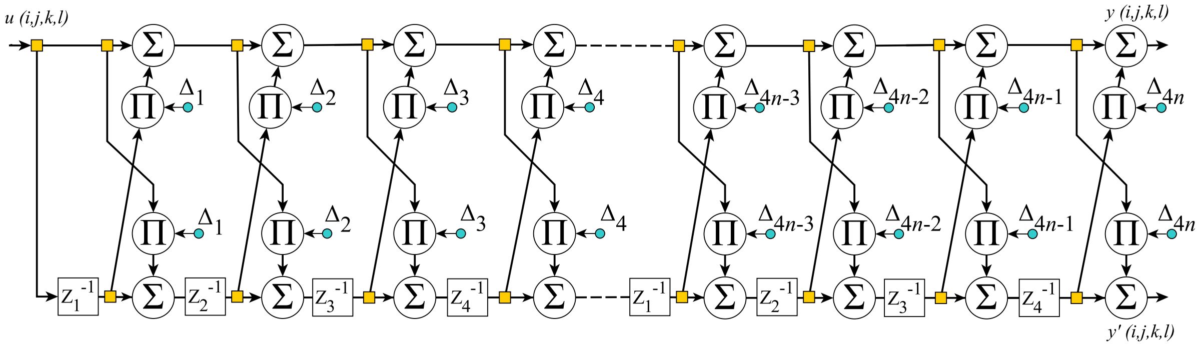

Fig. 1. Generalized 4D FIR digital filter with lattice structure

The subsequent goal is the derivation of the 4D generalized cyclic state–space model $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$ (1,2).

To formulate the state–space equations, for the 4D model, $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$, the step-by-step circuit–procedure follows:

1	Use the circuit representation, depicted in Fig. 1
2	Label the outputs of the delay elements $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$ that correspond to the states of the model
3	Write, by inspection, one state equation for every delay element $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$
4	Rearrange the equations to have blocks of the state variables: $\mathbf{x}^h$, $\mathbf{x}^v$, $\mathbf{x}^d$, $\mathbf{x}^t$
5	Generalize the results.

TABLE I

The resulting cyclic state–space model and its matrices are given below.

A. 4D Cyclic State–Space Model

$$\begin{aligned} \dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (6)\\ y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (7)\end{aligned}$$

where, $\dot{x}(i,j,k,l)$ and $x(i,j,k,l)$ are given in (3),(4), and

$$\mathbf{A} = \left[\ \mathbf{A}_1 \mid \mathbf{A}_2\ \right] \quad (8)$$

with,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \Delta_1\Delta_2 & 1 & 0 \\ \Delta_1\Delta_3 & \Delta_2\Delta_3 & \ddots \\ \cdots & \Delta_2\Delta_4 & \ddots \\ \cdots & \cdots & \cdots \\ \Delta_1\Delta_{4n-1} & \Delta_2\Delta_{4n-1} & \cdots \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} \cdots & \cdots & & 0 \\ \ddots & & & \vdots \\ & & & \vdots \\ \ddots & \ddots & & \vdots \\ 1 & 0 & \ddots & \vdots \\ \Delta_{4n-3}\Delta_{4n-2} & 1 & 0 & 0 \\ \cdots & \Delta_{4n-3}\Delta_{4n-2} & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{b} &= \left[\ 1\ \ \Delta_1\ \ \Delta_2\ \ \cdots\ \ \cdots\ \ \cdots\ \ \Delta_{4n-1}\ \right]' \\ \mathbf{c}' &= \left[\ \Delta_1\ \ \Delta_2\ \ \cdots\ \ \cdots\ \ \cdots\ \ \Delta_{4n}\ \right] \\ d &= 1 \end{aligned}$$

The dimensions of the above matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}', \mathbf{d}$ of the above 4D state–space model, have the following dimensions respectively: $4n \times 4n$, $4n \times 1$, $1 \times 4n$.

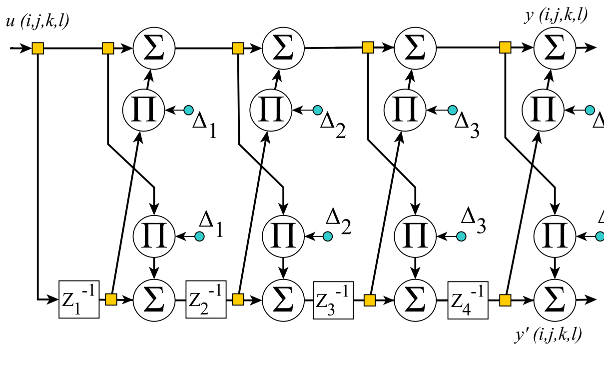

Fig. 2. First–order 4D digital filter with lattice structure

B. Example: First–order 4D FIR lattice discrete filter

Considering the output $y(i,j,k,l)$ of the first–order circuit realization given in Fig. 2., the corresponding 4D state–space realization takes on the following form:

$$\begin{aligned} \dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (9)\\ y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (10)\end{aligned}$$

where,

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l) \\ x_1^t(i,j,k,l+1) \end{bmatrix}$$

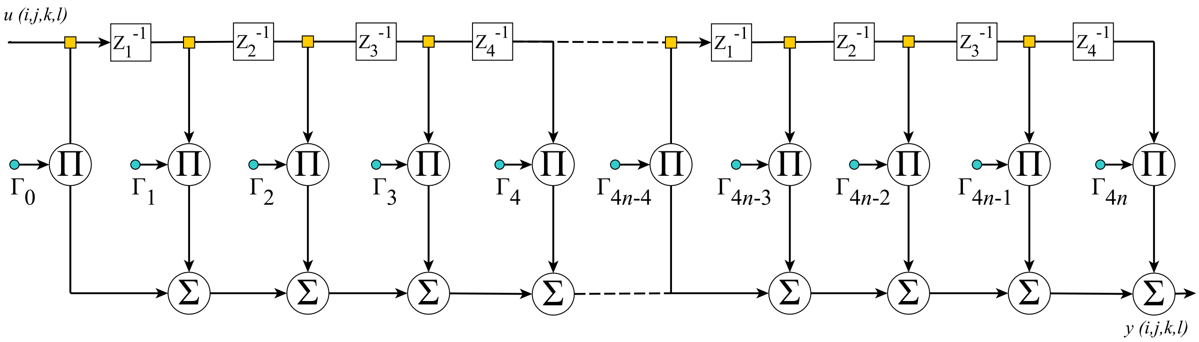

Fig. 3. Generalized 4D FIR direct–form digital filter

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l) \\ x_1^t(i,j,k,l) \end{bmatrix}$$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \Delta_1\Delta_2 & 1 & 0 & 0 \\ \Delta_1\Delta_3 & \Delta_2\Delta_3 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \\ \mathbf{c}' &= \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \end{bmatrix}, d = 1 \end{aligned}$$

Using (5) the corresponding 4D transfer function $T(z_1, z_2, z_3, z_4)$ of (9,10) is presented in the following table–transfer function:

Variables	FIR Coefficients
$z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}$	Δ_4
$z_1^{-1}z_2^{-1}z_3^{-1}$	Δ_3
$z_1^{-1}z_2^{-1}z_4^{-1}$	$\Delta_2\Delta_3\Delta_4$
$z_1^{-1}z_2^{-1}$	Δ_2
$z_1^{-1}z_3^{-1}z_4^{-1}$	$\Delta_1\Delta_2\Delta_4$
$z_1^{-1}z_3^{-1}$	$\Delta_1\Delta_2\Delta_3$
$z_1^{-1}z_4^{-1}$	$\Delta_1\Delta_3\Delta_4$
z_1^{-1}	Δ_1
$z_2^{-1}z_3^{-1}z_4^{-1}$	$\Delta_1\Delta_4$
$z_2^{-1}z_3^{-1}$	$\Delta_1\Delta_3$
$z_2^{-1}z_4^{-1}$	$\Delta_1\Delta_2\Delta_3\Delta_4$
z_2^{-1}	$\Delta_1\Delta_2$
$z_3^{-1}z_4^{-1}$	$\Delta_2\Delta_4$
z_3^{-1}	$\Delta_2\Delta_3$
z_4^{-1}	$\Delta_3\Delta_4$
const.	1

For $z_3^{-1} = z_4^{-1} = 0$, the corresponding 2D transfer function is, $\Delta_2 z_1^{-1} z_2^{-1} + \Delta_1 z_1^{-1} + \Delta_1\Delta_2 z_2^{-1} + 1$, as in [12].

It is noted that discrete filters with lattice structure, in the case where the multipliers or reflection coefficients are less than unity, have distinctive attributes related to its stability and also to its quantization properties [10].

IV. 4D DIRECT–FORM DIGITAL FILTER

A generalized 4D FIR direct–form structured digital filter is depicted in Fig. 3., following on the theory of 1D direct–form filters [10].

The proposed generalized circuit utilizes $4n$ delay elements, $8n$ multipliers and adders as shown in Fig. 3., where the number of both components are minimal.

The subsequent goal is the derivation of the 4D generalized cyclic state–space model $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$ (1,2).

To formulate the state–space equations, for the 4D model, $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$, the step-by-step circuit–procedure follows:

$$\dot{\mathbf{x}}(i,j,k,l) = \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (11)$$
$$y(i,j,k,l) = \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (12)$$

where, $\dot{x}(i,j,k,l)$ and $x(i,j,k,l)$ are given in (3),(4), and

$$\mathbf{A}_{(4n\times 4n)} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & & \vdots \\ 0 & 1 & 0 & & & \vdots \\ \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{b}_{(4n\times 1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{c}'_{(1\times 4n)} &= \begin{bmatrix} \Gamma_1 & \Gamma_2 & \cdots & \cdots & \cdots & \Gamma_{4n} \end{bmatrix} \\ d &= \Gamma_0 \end{aligned}$$

The matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}', \mathbf{d}$ of the above 4D state–space model, due to limited space, are given in Fig. 2 having the following dimensions respectively: $4n \times 4n$, $4n \times 1$, $1 \times 4n$.

A. Example: First–order 4D FIR direct–form digital filter

Considering the output $y(i,j,k,l)$ of the first–order circuit realization given in Fig. 4., the corresponding 4D state–space realization takes on the following form:

$$\dot{\mathbf{x}}(i,j,k,l) = \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (13)$$
$$y(i,j,k,l) = \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (14)$$

where,

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l) \\ x_1^t(i,j,k,l+1) \end{bmatrix}$$

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l) \\ x_1^t(i,j,k,l) \end{bmatrix}$$

with,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{c}' = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \end{bmatrix}$$
$$d = \Gamma_0$$

Using (5) the corresponding 4D transfer function $T(z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1})$ of (13,14) is presented in the following table–transfer function:

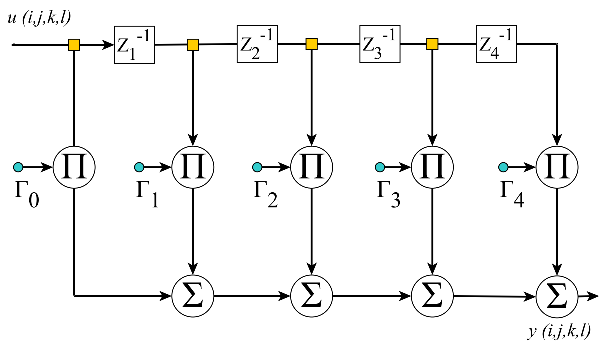

Fig. 4. First–order 4D FIR direct–form digital filter

Variables	FIR Coefficients
1	Γ_0
z_1^{-1}	Γ_1
$z_1^{-1}z_2^{-1}$	Γ_2
$z_1^{-1}z_2^{-1}z_3^{-1}$	Γ_3
$z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}$	Γ_4

or

$$T(z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}) = \Gamma_0 + \sum_{m=1}^{4}[\Gamma_m \times \prod_{i=1}^{m} z_i^{-1}]$$

It is evident that the above 4D filter transfer–function has an increasing dimensional order with respect to the z variables.

V. Conclusion

This paper discusses the circuit and state–space realizations for 4D FIR digital filters embodying lattice and direct–form structures. In the proposed realizations, the number of delay elements and the dimension of the state–space vector are absolutely minimal (4n). The number of multipliers/adders, of the circuit realizations, are $8n/4n$ and $4n+1/4n$ respectively. Both transfer functions are symmetric and recursive with respect to their dimension.

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