IEEE LASCAS2013
Cusco, PERU

Realization of 4D Lattice Structured Digital Filters

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***Abstract*—This paper presents a circuit realization for four–dimensional (4D) lattice discrete filters. The proposed 4D circuit realization requires, for its implementation, a minimum number of delay elements. Further, the dimension of the state-vector, of the derived 4D state space model, is minimal and its 4D transfer function is characterized by the all–pass property. A step-by-step low-order example is provided to demonstrate the proposed minimality of both, circuit, and state space realizations.**

I. Introduction

There is an on-going reinvigoration of interest for the study of multidimensional (nD) systems and filters; which has been motivated by the appealing nD theory and the variety of applications in the areas of digital signal and image processing, distributed parameter systems, geophysical sciences, intrusion detection, etc. [1]–[5].

Several contributions, in the emerging area of 4D filter/systems design, have recently been published. These publications include research works in filtering of 4D echocardiography data; object tracking in 4D state-space; 4D infinite impulse response filters; and usability of meteorological and oceanographic data for military applications [6]–[14]. These modern contributions emphasize the utilization of higher order data sets.

In this paper the problem of circuit and minimal state space realization for four–dimensional (4D) lattice discrete filters is considered. The need to provide minimal realization arises not only out of hardware requirements but also because sometimes non-minimal realizations often cause theoretical or computational difficulties due to the absence of the fundamental theorem of algebra for polynomials with more than one-dimension [4],[5].

Generalized minimal state space realizations have been reported for up to three–dimensions in continued fraction expandable systems, all–pole and all–zero filters, separable all–pass filters, separable and factorable, lattice filters, and reverse-lattice filters [15]–[19]. Discrete filters with lattice structure, in the case where the multiplier or reflection coefficients are less than unity, have distinctive attributes related to its stability and also to its quantization properties [20].

II. 4D Circuit and State Space Realizations

A generalized 4D digital filter with lattice structure is depicted in Fig. 1., following on the theory of 1D and 2D lattice filters [15],[20].

The proposed generalized circuit utilizes $4n$ delay elements, and $4n$ multipliers, as shown in Fig. 1., where the number of both components are minimal. For a single 4D section, where $n = 1$ in Fig. 2., the number of delay elements and multipliers are four, which is minimal.

The subsequent goal is the derivation of the 4D generalized state space model $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$ having the well known Givone-Roeeser structure, extended to 4D, with Cyclic dimensional arrangement with respect to the four independent variables: $x_1^h(i,j,k,l), x_1^v(i,j,k,l), x_1^d(i,j,k,l), x_1^t(i,j,k,l)$ [21],[22]. To acquire the state-space equations for the 4D model, $(\mathbf{A}, \mathbf{b}, \mathbf{c}', d)$, the following procedure is applied:

- Use the circuit representation, depicted in Fig. 1
- Label the outputs of the delay elements $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$ that correspond to the states of the model
- Write, by inspection, one state equation for every delay element $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$
- Rearrange the equations to have blocks of the state variables: $\mathbf{x}^h$, $\mathbf{x}^v$, $\mathbf{x}^d$, $\mathbf{x}^t$
- Generalize the results.

Following the above procedure the generalized 4D state space matrix–vectors $\{\mathbf{A}, \mathbf{b}, \mathbf{c}', d\}$ and the scalar d of (1), are derived. The overall 4D Givone-Roesser state space system–model having cyclic structure, with respect to the variables, is given below.

$$
\begin{aligned}
\dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \qquad (1)\\
y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l)
\end{aligned}
$$

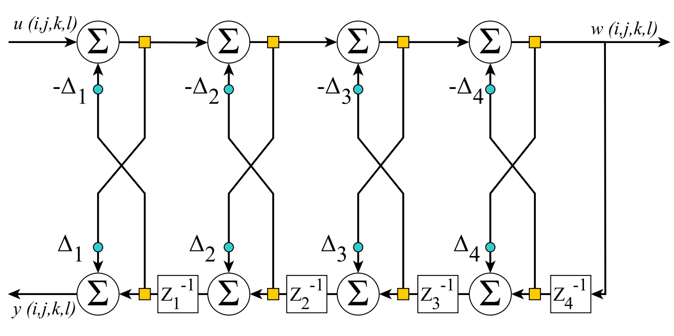

Fig. 2. A single 4D section

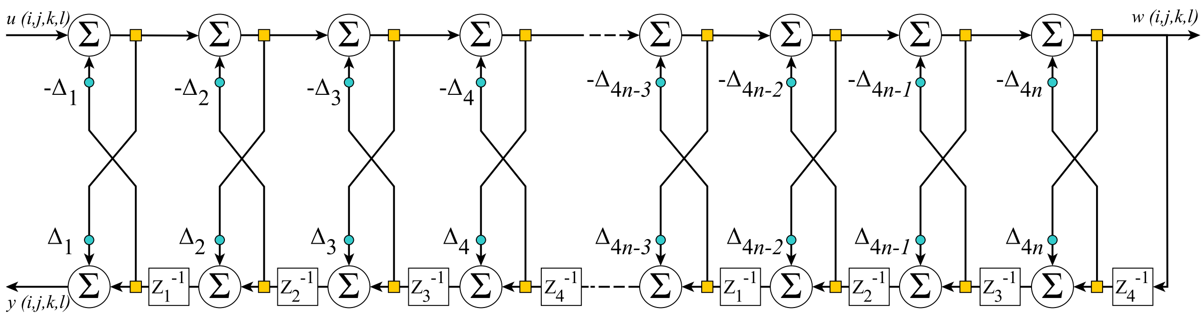

Fig. 1. Generalized 4D digital filter with lattice structure

where,

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l)) \\ x_1^t(i,j,k,l) \\ \cdots \\ x_{4n}^h(i,j,k,l) \\ x_{4n}^v(i,j,k,l) \\ x_{4n}^d(i,j,k,l)) \\ x_{4n}^t(i,j,k,l) \end{bmatrix}$$

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l)) \\ x_1^t(i,j,k,l+1) \\ \cdots \\ x_{4n}^h(i+1,j,k,l) \\ x_{4n}^v(i,j+1,k,l) \\ x_{4n}^d(i,j,k+1,l)) \\ x_{4n}^t(i,j,k,l+1) \end{bmatrix}$$

The matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}', \mathbf{d}$ of the above 4D state space model, due to limited space, are given in Fig. 2 having the following dimensions respectively: $4n \times 4n,\ 4n \times 1,\ 1 \times 4n$.

Applying the 4D z-transform on (1), its corresponding 4D transfer-function takes the following form:

$$H(z_1, z_2, z_3, z_4) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b} \tag{2}$$

where, $\mathcal{Z} = z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n \oplus z_3\mathbf{I}_n \oplus z_4\mathbf{I}_n$ with $\oplus$ denoting the direct sum.

The procedure to facilitate the traversal between the state space models of Givone-Roesser and Cyclic, both extended to 4D, is provided in Ref. [22].

III. Examples

A. First–order 4D All–Pass Lattice filter

Considering the output of the first-order circuit realization given in Fig. 2 to be $y(i,j,k,l)$, the corresponding 4D state space realization takes on the following form:

$$\begin{aligned} \dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \\ y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \end{aligned} \tag{3}$$

where,

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l) \\ x_1^t(i,j,k,l+1) \end{bmatrix}$$

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l) \\ x_1^t(i,j,k,l) \end{bmatrix}$$

with,

$$\mathbf{A} = \begin{bmatrix} -\Delta_1\Delta_2 & 1-\Delta_2^2 & 0 & 0 \\ -\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 \\ -\Delta_1\Delta_4 & -\Delta_2\Delta_4 & -\Delta_3\Delta_4 & 1-\Delta_4^2 \\ -\Delta_1 & -\Delta_2 & -\Delta_3 & -\Delta_4 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ 1 \end{bmatrix}$$

$$\mathbf{c}' = \begin{bmatrix} 1-\Delta_1^2 & 0 & 0 & 0 \end{bmatrix}$$

$$d = \Delta_1$$

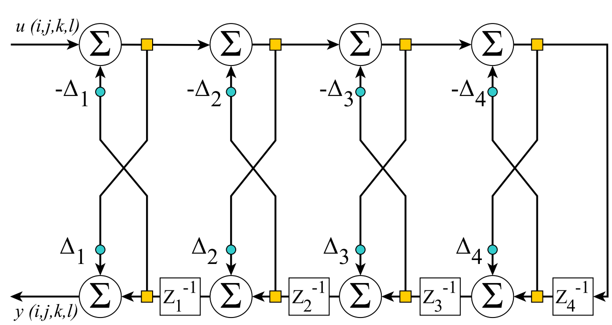

Fig. 4. First–order 4D All–Pass Lattice filter

$$
\begin{aligned}
\mathbf{A} &= \left[\begin{array}{cccccc}
-\Delta_1\Delta_2 & 1-\Delta_2^2 & \cdots & 0 & 0 & 0 \\
-\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 & 0 & 0 \\
-\Delta_1\Delta_4 & -\Delta_2\Delta_4 & 1-\Delta_3^2 & 1-\Delta_4^2 & 0 & 0 \\
\cdots & \cdots & \ddots & \ddots & \cdots & 0 \\
-\Delta_1\Delta_{4n-1} & \cdots & \cdots & -\Delta_{4n-2}\Delta_{2n-1} & \ddots & \\
-\Delta_1\Delta_{4n} & -\Delta_2\Delta_{4n} & \cdots & -\Delta_{4n-2}\Delta_{2n} & -\Delta_{4n-1}\Delta_{4n} & 1-\Delta_{4n}^2 \\
-\Delta_1 & -\Delta_2 & \cdots & -\Delta_{4n-2} & -\Delta_{4n-1} & -\Delta_{4n}
\end{array}\right] \\
\mathbf{b} &= \left[\begin{array}{c}
\Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \cdots \\ \Delta_{4n-1} \\ \Delta_{4n} \\ 1
\end{array}\right] \\
\mathbf{c}' &= \left[\begin{array}{cccccc} 1-\Delta_1^2 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \\
d &= \Delta_1
\end{aligned}
$$

Fig. 3. 4D state space matrix–vectors $\mathbf{A}, \mathbf{b}, \mathbf{c}'$ and the scalar d, of (3)

Using (2) the corresponding 4D transfer function $T(z_1, z_2, z_3, z_4)$ of (3) is presented in the following table–transfer function:

Variables	Numerator	Denominator
constant	1	Δ_1
z_4	Δ_4	$\Delta_1\Delta_4$
z_3	$\Delta_3\Delta_4$	$\Delta_1\Delta_3\Delta_4$
z_3z_4	Δ_3	$\Delta_1\Delta_3$
z_2	$\Delta_2\Delta_3$	$\Delta_1\Delta_2\Delta_3$
z_2z_4	$\Delta_2\Delta_3\Delta_4$	$\Delta_1\Delta_2\Delta_3\Delta_4$
z_2z_3	$\Delta_2\Delta_4$	$\Delta_1\Delta_2\Delta_4$
$z_2z_3z_4$	Δ_2	$\Delta_1\Delta_2$
z_1	$\Delta_1\Delta_2$	Δ_2
z_1z_4	$\Delta_1\Delta_2\Delta_4$	$\Delta_2\Delta_4$
z_1z_3	$\Delta_1\Delta_2\Delta_3\Delta_4$	$\Delta_2\Delta_3\Delta_4$
$z_1z_3z_4$	$\Delta_1\Delta_2\Delta_3$	$\Delta_2\Delta_3$
z_1z_2	$\Delta_1\Delta_3$	Δ_3
$z_1z_2z_4$	$\Delta_1\Delta_3\Delta_4$	$\Delta_3\Delta_4$
$z_1z_2z_3$	$\Delta_1\Delta_4$	Δ_4
$z_1z_2z_3z_4$	Δ_1	1

It is noted that in the above 4D transfer function–table, the denominator polynomial is the mirror–image polynomial of the numerator and vice–versa as in the case of 1D all–pass discrete filters [20].

B. First–order 4D All–Pole Lattice filter

Considering the output of the first-order circuit realization given in Fig.4 to be $y(i, j, k, l)$, the corresponding 4D state space realization takes on the following form:

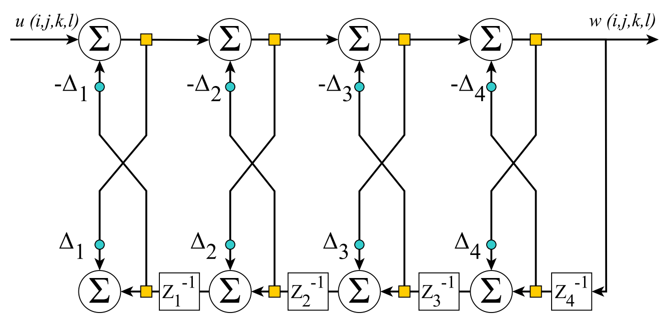

Fig. 5. First–order All-Pole Lattice filter

$$
\begin{aligned}
\dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \qquad (4) \\
y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l)
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{A} &= \left[\begin{array}{cccc}
-\Delta_1\Delta_2 & 1-\Delta_2^2 & 0 & 0 \\
-\Delta_1\Delta_3 & -\Delta_2\Delta_3 & 1-\Delta_3^2 & 0 \\
-\Delta_1\Delta_4 & -\Delta_2\Delta_4 & -\Delta_3\Delta_4 & 1-\Delta_4^2 \\
-\Delta_1 & -\Delta_2 & -\Delta_3 & -\Delta_4
\end{array}\right] \\
\mathbf{b} &= \left[\begin{array}{c} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ 1 \end{array}\right] \\
\mathbf{c}' &= \left[\begin{array}{cccc} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \end{array}\right]
\end{aligned}
$$

Using (2) the corresponding 4D transfer function $T(z_1, z_2, z_3, z_4)$ of (4) is presented in the following table–transfer function:

Variables	Numerator	Denominator
constant	1	Δ_1
z_4	0	$\Delta_1\Delta_4$
z_3	0	$\Delta_1\Delta_3\Delta_4$
z_3z_4	0	$\Delta_1\Delta_3$
z_2	0	$\Delta_1\Delta_2\Delta_3$
z_2z_4	0	$\Delta_1\Delta_2\Delta_3\Delta_4$
z_2z_3	0	$\Delta_1\Delta_2\Delta_4$
$z_2z_3z_4$	0	$\Delta_1\Delta_2$
z_1	0	Δ_2
z_1z_4	0	$\Delta_2\Delta_4$
z_1z_3	0	$\Delta_2\Delta_3\Delta_4$
$z_1z_3z_4$	0	$\Delta_2\Delta_3$
z_1z_2	0	Δ_3
$z_1z_2z_4$	0	$\Delta_3\Delta_4$
$z_1z_2z_3$	0	Δ_4
$z_1z_2z_3z_4$	0	1

It is conspicuous that the above 4D filter transfer-function has the basic characteristic of an all–pole discrete 4D filter.

IV. Conclusion

This paper discusses the circuit and state-space realizations for 4D digital filters having a lattice structure. In the proposed realizations the number of delay elements and the dimension of the state-space vector are absolutely minimal $(4n)$, and the number of multipliers and adders of the circuit realization is $8n$. The 4D transfer functions of the realizations are characterized by the all-pass property, as in the classical digital signal processing filtering, and it is evident from examples provided.

V. Acknowledgments

The authors would like to thank all the reviewers for their valuable comments.

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