Circuit and State-Space Realization of 2-D All-Pass Digital Filters via the Bilinear Transform

S. J. Varoufakis and G. E. Antoniou

Abstract—**In this paper the problem of the circuit and state-space realization of first- and second-order two-dimensional (2-D) all-pass digital filters is considered. The method uses the 2-D bilinear transform. The multipliers and delay elements required for the realization equals the maximum number of the numerator or denominator coefficients of the given filter. The matrices A, b, and c^t and the scalar d of the Givone–Roesser state space model are derived, based on the circuit realization, for first- and second-order all-pass digital filters.**

Key Words—**Digital filters, state space, modeling, realization, matrix theory.**

I. Introduction

Recently, there has been an increasing interest in the study of two-dimensional (2-D) all-pass digital filters (APDF). These filters may be used in transforming 1-D IIR filters to 2-D IIR filters. First- and second-order 2-D APDF can be used in cascade in order to obtain filters of higher order. Also these filters are useful for the phase linearization of 2-D IIR filters [1]–[10].

The problem of the realization with or without some optimal criteria is an important factor in the analysis and design of digital filters. For the case of 2-D APDF a number of realizations have been proposed by various authors. Joshi *et al.* first studied the problem of first-order 2-D all-pass network realization [1]. Ganapathy *et al.* proposed a method to realize first-order 2-D APDF with two delays and six multipliers [2]. Sudhakara *et al.* reported a realization structure for first-order 2-D APDF with two delays and six multipliers [3]. Manivannan *et al.* described a method for the realization of 2-D APDF with three multipliers and three delay elements [4]–[6]. Kwan proposed a method for realizing first-order 2-D all-pass–all-pole digital filters [7]. Reddy *et al.* reported a method for the realization of separable first- and second-order 2-D APDF with minimal multipliers and delays [8]. Recently the orthogonal realization of first-order all-pass filters for 2-D systems was studied in [9].

In this paper the problem of the circuit and state space realization for first- and second-order 2-D APDF, using the bilinear transform, is presented. The matrices A, b, and c^t and the scalar d of the state-space model of Givone–Roesser type are derived by inspection from the circuit representation of the 2-D APDF. The bilinear transform, in two dimensions, has been used for the study of stability, transformation to state-space systems, overflow oscillations, etc. [11], [12].

II. Bilinear Transform

A useful mathematical tool in system and signal analysis is the bilinear transform, which is defined as follows [13], [14]:

$$z = \frac{1 + wT/2}{1 - wT/2} \tag{1}$$

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or, solving for w,

$$w = (2/T)\frac{z-1}{z+1} \tag{2}$$

where T is the sampling period given by

$$2/T = \frac{\omega_s}{\pi}$$

or

$$2/T = \frac{\lambda}{\pi}\omega_c$$

where, $\omega_s = \lambda\omega_c$, ω_s denotes the sampling frequency, ω_c the cutoff frequency, and λ is a constant.

Equation (2) can also be written as

$$w = k\frac{z-1}{z+1} \tag{3}$$

where $k = (\lambda/\pi)\omega_c$.

Equivalently, (3) can be written as

$$z = \frac{1 + w/k}{1 - w/k}. \tag{4}$$

The bilinear transform in two dimensions is given by [11], [12]:

$$z_1 = \frac{1 + w_1/k_1}{1 - w_1/k_1} \Rightarrow z_1^{-1} = \frac{k_1 w_1^{-1} - 1}{k_1 w_1^{-1} + 1} \tag{5a}$$

and

$$z_2 = \frac{1 + w_2/k_2}{1 - w_2/k_1} \Rightarrow z_1^{-1} = \frac{k_2 w_2^{-1} - 1}{k_2 w_2^{-1} + 1}. \tag{5b}$$

Note that through the above bilinear transform the unit circle of the z plane transforms to the imaginary axis of the w plane. Also, it is remarked that due to nonlinearity and the frequency warping inherent to the bilinear transform, one has to preserve the initial and the transformed transfer function. This can be done by increasing the sampling frequency or by introducing a normalization factor k to compensate the magnitude response [14]. The warping effect can be alleviated "by prewarp" of the critical frequencies [13].

III. Realization of First-Order 2-D APDF

Consider the first-order 2-D APDF, described by the following transfer function:

$$H(z_1^{-1}, z_2^{-1}) = \frac{\sum_{i=0}^{1}\sum_{j=0}^{1} a_{1-i,1-j} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j} z_1^{-i} z_2^{-j}}, \quad \text{with } a_{00} = 1 \tag{6}$$

or

$$H(z_1^{-1}, z_2^{-1}) = \frac{a_{11} + a_{10}z_2^{-1} + a_{01}z_1^{-1} + z_1^{-1}z_2^{-1}}{1 + a_{01}z_2^{-1} + a_{10}z_1^{-1} + a_{11}z_1^{-1}z_2^{-1}}.$$

The above transfer function can be written in matrix form as

$$H(z_1^{-1}, z_2^{-1}) = \frac{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{10} \\ a_{01} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}. \tag{7}$$

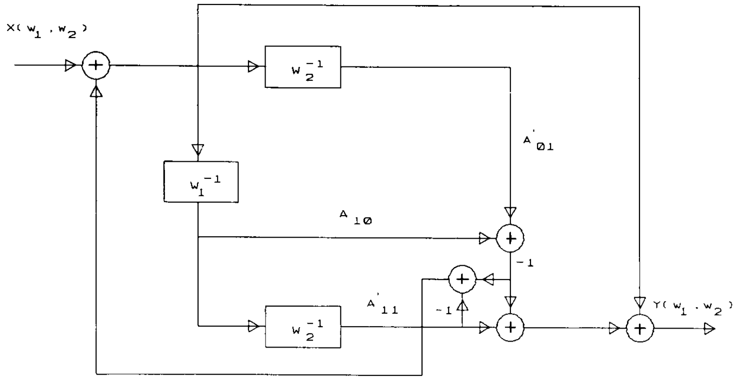

Fig. 1. Circuit realization of first-order 2-D APDF (14).

Applying the 2-D bilinear transform on the transfer function (7), the resulting transfer function is

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}1 & (k_1w_1^{-1}-1)/(k_1w_1^{-1}+1)\end{bmatrix}\begin{bmatrix}a_{11} & a_{10}\\ a_{01} & 1\end{bmatrix}\cdot\begin{bmatrix}1\\ (k_2w_2^{-1}-1)/(k_2w_2^{-1}+1)\end{bmatrix}}{\begin{bmatrix}1 & (k_1w_1^{-1}-1)/(k_1w_1^{-1}+1)\end{bmatrix}\begin{bmatrix}1 & a_{01}\\ a_{10} & a_{11}\end{bmatrix}\cdot\begin{bmatrix}1\\ (k_2w_2^{-1}-1)/(k_2w_2^{-1}+1)\end{bmatrix}}. \tag{8}$$

Furthermore, the transfer function (8) can be written as follows:

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}(k_1w_1^{-1}+1) & (k_1w_1^{-1}-1)\end{bmatrix}\begin{bmatrix}a_{11} & a_{10}\\ a_{01} & 1\end{bmatrix}\begin{bmatrix}(k_2w_2^{-1}+1)\\ (k_2w_2^{-1}-1)\end{bmatrix}}{\begin{bmatrix}(k_1w_1^{-1}+1) & (k_1w_1^{-1}-1)\end{bmatrix}\begin{bmatrix}1 & a_{01}\\ a_{10} & a_{11}\end{bmatrix}\begin{bmatrix}(k_2w_2^{-1}+1)\\ (k_2w_2^{-1}-1)\end{bmatrix}} \tag{9}$$

or

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1 & -1\\ k_1 & k_1\end{bmatrix}\begin{bmatrix}a_{11} & a_{10}\\ a_{01} & 1\end{bmatrix}\begin{bmatrix}1 & k_2\\ -1 & k_2\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1 & -1\\ k_1 & k_1\end{bmatrix}\begin{bmatrix}1 & a_{01}\\ a_{10} & a_{11}\end{bmatrix}\begin{bmatrix}1 & k_2\\ -1 & k_2\end{bmatrix}\begin{bmatrix}1 & w_2^{-1}\end{bmatrix}}. \tag{10}$$

Moreover, multiplying the middle two matrices on the left, of the numerator and denominator, of (10), yields

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}-a_{01}+a_{11} & -1+a_{10}\\ k_1(a_{01}+a_{11}) & k_1(1+a_{10})\end{bmatrix}\begin{bmatrix}1 & k_2\\ -1 & k_2\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1-a_{10} & a_{01}-a_{11}\\ k_1(1+a_{10}) & k_1(a_{01}+a_{11})\end{bmatrix}\begin{bmatrix}1 & k_2\\ -1 & k_2\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}. \tag{11}$$

Then, multiplying the middle two matrices, we have

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}-a_{01}-a_{10}+a_{11}+1 & k_2(-a_{01}+a_{10}+a_{11}-1)\\ k_1(1-a_{01}-a_{10}+a_{11}) & k_2(1+a_{01}+a_{10}+a_{11})\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1-a_{01}-a_{10}+a_{11} & k_2(1+a_{01}-a_{10}-a_{11})\\ k_1(1-a_{01}+a_{10}-a_{11}) & k_1k_2(1+a_{01}+a_{10}+a_{11})\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}. \tag{12}$$

Let us now define

$$A_{00}=1-a_{01}-a_{10}+a_{11}\neq 0$$
$$A_{01}=(1+a_{01}-a_{10}-a_{11})k_2$$
$$A_{10}=(1-a_{01}+a_{10}-a_{11})k_1$$
$$A_{11}=(1+a_{01}+a_{10}+a_{11})k_1k_2.$$

Since A_{00} is defined to be nonzero, (12) can be written as

$$H(w_1^{-1},w_2^{-1})=\frac{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1 & -A'_{01}\\ -A'_{10} & A'_{11}\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}}{\begin{bmatrix}1 & w_1^{-1}\end{bmatrix}\begin{bmatrix}1 & A'_{01}\\ A'_{10} & A'_{11}\end{bmatrix}\begin{bmatrix}1\\ w_2^{-1}\end{bmatrix}} \tag{13}$$

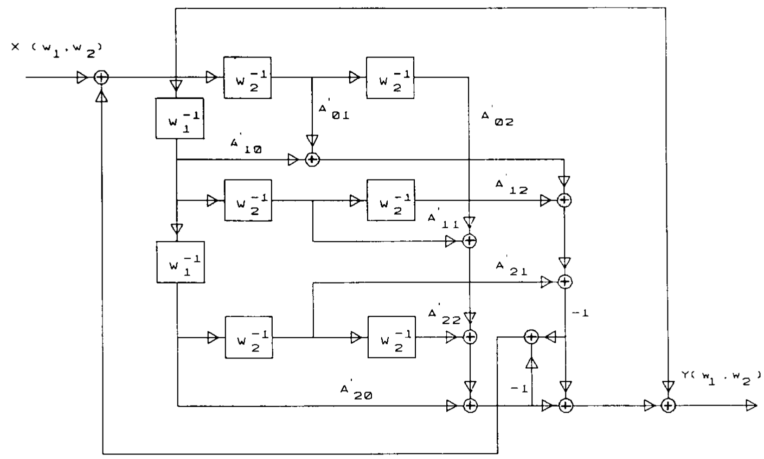

Fig. 2. Circuit realization of the second-order 2-D APDF (18).

where

$$A'_{01} = A_{01}/A_{00}$$
$$A'_{10} = A_{10}/A_{00}$$
$$A'_{11} = A_{11}/A_{00}.$$

Finally, the transfer function (7), using the bilinear transform, can be written as

$$H(w_1^{-1}, w_2^{-1}) = \frac{1 - A'_{10}w_1^{-1} - A'_{01}w_2^{-1} + A'_{11}w_1^{-1}w_2^{-1}}{1 + A'_{10}w_1^{-1} + A'_{01}w_2^{-1} + A'_{11}w_1^{-1}w_2^{-1}}. \quad (14)$$

The circuit implementation of (14), is depicted in Fig. 1.

Note that the realization, given in Fig. 1, employs three delay elements and three multipliers.

IV. Realization of Second-Order 2-D APDF

Consider the second-order 2-D APDF, described by the transfer function

$$H(z_1^{-1}, z_2^{-1}) = \frac{B(z_1^{-1}, z_2^{-1})}{A(z_1^{-1}, z_2^{-1})} \quad (15)$$

where the numerator and denominator polynomials $B(z_1^{-1}, z_2^{-1})$ and $A(z_1^{-1}, z_2^{-1})$ have the following forms:

$$B(z_1^{-1}, z_2^{-1}) = \begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} \end{bmatrix} \begin{bmatrix} a_{22} & a_{21} & a_{20} \\ a_{12} & a_{11} & a_{10} \\ a_{02} & a_{01} & a_{00} \end{bmatrix} \begin{bmatrix} 1 \\ z_2^{-1} \\ z_2^{-2} \end{bmatrix}$$
$$= \sum_{i=0}^{2} \sum_{j=0}^{2} a_{2-i,2-j} z_1^{-i} z_2^{-j} \quad (16a)$$

and

$$A(z_1^{-1}, z_2^{-1}) = \begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ z_2^{-1} \\ z_2^{-2} \end{bmatrix}$$
$$= \sum_{i=0}^{2} \sum_{j=0}^{2} a_{i,j} z_1^{-i} z_2^{-j} \quad (16b)$$

where $a_{00} = 1$.

Applying the 2-D bilinear transform to (16a) and (16b) and after some matrix manipulations, as in the case of the first order, the APDF (15) yields

$$H(w_1^{-1}, w_2^{-1}) = \frac{\begin{bmatrix} 1 & w_1^{-1} & w_1^{-2} \end{bmatrix} \begin{bmatrix} 1 & -A'_{01} & A'_{02} \\ -A'_{10} & A'_{11} & -A'_{12} \\ A'_{20} & -A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} 1 \\ w_2^{-1} \\ w_2^{-2} \end{bmatrix}}{\begin{bmatrix} 1 & w_1^{-1} & w_1^{-2} \end{bmatrix} \begin{bmatrix} 1 & A'_{01} & A'_{02} \\ A'_{10} & A'_{11} & A'_{12} \\ A'_{20} & A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} 1 \\ w_2^{-1} \\ w_2^{-2} \end{bmatrix}}. \quad (17)$$

Finally the transfer function (15), using the bilinear transform, can be written as

$$H(w_1^{-1}, w_2^{-1}) = \frac{\sum_{i=0}^{2} \sum_{i=0}^{2} A'_{ij}(-1)^{i+j} w_1^{-i} w_2^{-j}}{\sum_{i=0}^{2} \sum_{j=0}^{2} A'_{ij} w_1^{-i} w_2^{-j}}, \qquad A'_{00} = 1 \quad (18)$$

where

$$A'_{01} = A_{01}/A_{00},\ A'_{02} = A_{02}/A_{00}$$
$$A'_{10} = A_{10}/A_{00},\ A'_{11} = A_{11}/A_{00},\ A'_{12} = A_{12}/A_{00}$$
$$A'_{20} = A_{20}/A_{00},\ A'_{21} = A_{21}/A_{00},\ A'_{22} = A_{22}/A_{00}$$

with

$$A_{00} = 1 - a_{01} + a_{02} - a_{10} + a_{11} - a_{12} + a_{20} - a_{21} + a_{22} \neq 0$$
$$A_{01} = 2(1 - a_{02} - a_{10} + a_{12} + a_{20} - a_{22})k_2$$
$$A_{02} = (1 + a_{01} + a_{02} - a_{10} - a_{11} - a_{12} + a_{20} + a_{21} + a_{22})k_2^2$$
$$A_{10} = 2(1 - a_{01} + a_{02} - a_{20} + a_{21} - a_{22})k_1$$
$$A_{11} = 4(1 - a_{02} - a_{20} + a_{22})k_1 k_2$$
$$A_{12} = 2(1 + a_{01} + a_{02} - a_{20} - a_{21} - a_{22})k_1 k_2^2$$
$$A_{20} = 1 - a_{01} + a_{02} + a_{10} - a_{11} + a_{12} + a_{20} - a_{21} + a_{22})k_1^2$$
$$A_{21} = 2(1 + a_{10} - a_{02} + a_{20} - a_{12} - a_{22})k_1^2 k_2$$
$$A_{22} = (1 + a_{01} + a_{02} + a_{10} + a_{11} + a_{12} + a_{20} + a_{21} + a_{22})k_1^2 k_2^2.$$

The circuit implementation of (18) is depicted in Fig. 2. Note that this realization employs eight delay elements and eight multipliers.

Note: The delay elements are arranged in a matrix form, which is suggested from the coefficients matrices. The numerator and denominator matrices consist of elements with equal absolute values. The number of delay elements as well as the number of the multipliers correspond to the maximum number of either the numerator or the denominator coefficients. In that respect the proposed circuit configuration has the minimum number of multipliers and delay elements. Similar results, with respect to the number of delay elements and multipliers, have been reported in [4].

V. State-Space Realization

From the circuit realizations given in Figs. 1 and 2, 2-D state-space representations can be derived. To this end, let us assume that the outputs of the delay elements w_1^{-1} and w_2^{-1} of Fig. 1 and 2 correspond to the states of the following 2-D state-space model of Givone–Roesser type [15]:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + bu(i,j)$$

$$y(i,j) = c^t \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + du(i,j). \tag{19}$$

Then we can write one state equation for every delay element; after some algebraic manipulations, we conclude that the matrices A, b, c^t, and the scalar d of the above model (19) are as follows.

Case 1: For the case of first-order filters the state-space realization has the form (19), where

$$A = \left[\begin{array}{c|cc} -A'_{10} & -A'_{01} & -A'_{11} \\ \hline -A'_{10} & -A'_{01} & -A'_{11} \\ 1 & 0 & 0 \end{array}\right], \quad b = \left[\begin{array}{c} 1 \\ \hline 1 \\ 0 \end{array}\right]$$

$$c^t = \left[\begin{array}{cc|c} -2A'_{10} & -2A'_{01} & 0 \end{array}\right] \quad \text{and} \quad d = 1. \tag{20}$$

Case 2: For the case of second-order filters the state-space realization has the form (19), where

$$A = \left[\begin{array}{cc|cccccc} -A'_{10} & -A'_{20} & -A'_{01} & -A'_{11} & -A'_{21} & -A'_{02} & -A'_{12} & -A'_{22} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -A'_{10} & -A'_{20} & -A'_{01} & -A'_{11} & -A'_{21} & -A'_{02} & -A'_{12} & -A'_{22} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad b = \left[\begin{array}{c} 1 \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$c^t = \left[\begin{array}{cc|cccccc} -2A'_{10} & 0 & -2A'_{01} & 0 & -2A'_{21} & 0 & -2A'_{12} & 0 \end{array}\right] \quad \text{and} \quad d = 1.$$

The above state-space realizations, (20) and (21), represent the transformed transfer functions of the initial digital filters (7) and (15), using the bilinear transform. Since it has been shown in [16] that the application of the bilinear transform might result in an unstable transfer function, the stability of these new systems (20) and (21) has to be studied.

VI. Conclusions

A method for the realization of first- and second-order 2-D all-pass digital filters has been presented. Using this method, which is based on the 2-D bilinear transform, one can readily realize a) first-order 2-D APDF with three multipliers and three delay elements, and b) second-order APDF with eight multipliers and eight delay elements. For the further study of these systems, using state-space techniques, the corresponding state space realizations in the Givone–Roesser setting have been derived.

References

[1] R. C. Joshi, H. Singh, and S. Rai, "First order all-pass network realization," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1089–1091, 1981.

[2] V. Ganapathy, D. R. Reddy, and P. S. Reddy, "Minimal delay realization of first order 2-D all-pass digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 1577–1579, 1983.

[3] M. S. Reddy, S. C. D. Roy, and S. N. Hazra, "Realization of first-order 2-D all-pass digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 1011–1013, 1986.

[4] K. Manivannan and C. Eswaran, "Minimal multiplier realization of all-pass digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-35, pp. 480–482, 1988.

[5] P. S. Reddy and M. N. S. Swamy, "Comments on minimal multiplier realization of all-pass digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-36, pp. 1147–1148, 1989.

[6] T. Venkateswarlu and C. Eswaran, "Comments on minimal multiplier realization of all-pass digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-37, pp. 670–672, 1990.

[7] H. K. Kwan, "New realizations of first-order 2-D all-pass and all-pole digital filters," *Electron. Lett.*, vol. 24, pp. 224–226, 1988.

[8] P. S. Reddy, P. Satyanarayana, and M. N. S. Swamy, "Realization of separable first- and second-order 2-D all-pass digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-36, pp. 1146–1147, 1989.

[9] J. V. Gisladottir, H. Lev-Ari, and T. Kailath, "Orthogonal realization of first-order all-pass filters for two-dimensional signals," *Int. J. Multidimensional Syst. Signal Processing*, vol. 1, pp. 39–50, 1990.

[10] M. E. Zervakis and A. N. Venetsanopoulos, "Three-dimensional rotated digital filters: Design, stability, and applications," *Circuits Syst. Signal Processing*, vol. 9, pp. 383–408, 1990.

[11] J. H. Lodge and M. M. Fahmy, "The bilinear transformation of two-dimensional state-space systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 500–502, 1982.

[12] ——, "Stability and overflow oscillations in 2-D state-space digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1161–1171, 1981.

[13] A. Antoniou, *Digital Filters: Analysis and Design*. New York: McGraw Hill, 1979.

[14] M. T. Jong, *Methods of Discrete Signal and System Analysis*. New York: McGraw Hill, 1982.

[15] D. D. Givone and R. P. Roesser, "Multidimensional linear iterative circuits-general properties," *IEEE Trans. Comput.*, vol. C-21, 1972.

[16] D. Goodman, "Some difficulties with the bouble bilinear transform in 2-D recursive filter design," *Proc. IEEE*, vol. 65, pp. 796–797, 1978.

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